Competitive Information Design in Sequential Search

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Abstract

Advertisements often strategically disclose information to consumers who make decisions on further information acquisition and eventual purchase. Anderson and Renault (2006) model this problem using an information design framework, where the advertiser acts as a sender and the consumer as a receiver. We extend this model to a competitive setting with horizontally differentiated senders competing for a unit-demand receiver. Under costly inspection, the receiver's optimal sequential search action is given by Weitzman's Index Algorithm. We give a method, based on duality arguments, to verify whether a sender's given strategy constitutes a best response against his competitors (other senders). We establish the existence of an equilibrium in the game among senders when the prior distributions have no mass; we also illustrate that such equilibria may exhibit intricate behaviors. Finally, we meticulously characterize symmetric equilibria played by the senders for cases when the prior distributions have monotone increasing densities, while offering economic intuitions behind the insightful equilibrium structure.

1 Introduction

Consumers are often faced with horizontally differentiated products without being fully informed of the attributes offered by different brands. This information gap is often bridged, at least partially, by the advertisements that brands post to compete for the purchase. This paper studies both a brand's advertising strategy, viewed as an information-revealing process, in face of competition, and the game that engages multiple advertising brands.

Advertisement attracts consumers in multiple ways (Nelson, 1970; Milgrom and Roberts, 1986). We take as our starting point an influential model of *informative advertising* by Anderson and Renault (2006), where an advertisement is seen as a signal sent by the seller to directly – though perhaps only partially – inform a potential buyer about her valuation of the product. The signal, bound by legal sanctions, is credible. The unit-demand buyer, with posterior estimates from the advertisements, searches for a purchase.

In this context, advertisements are only meaningful when there is search friction. Anderson and Renault assume the buyer incurs an inspection cost in order to learn her value for any product

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and study a monopoly seller. In this work we study a market with multiple sellers. One immediate implication of this generalization is that the buyer's search is considerably more involved. Instead of having only two decisions to make — whether to inspect and whether to buy — the buyer now must decide in what order to inspect and when to buy (if at all). This sequential search problem for the buyer, modeled and solved by Weitzman (1978) as the celebrated Pandora Box problem, admits an optimal *Index Algorithm* . We assume in our model that the buyer adopts this algorithm.

In Anderson and Renault (2006), the seller strategically sets a price and commits to an advertising signaling scheme. As a first step toward understanding competitive advertising, we abstract away the pricing part and only consider strategic signaling: the buyer's value for each product is normalized to a range of $[0, 1]$; a seller gets utility 1 when the buyer makes a purchase from him, and 0 otherwise. This simplification is for two reasons: (1) The multi-sender scenario is more complicated to begin with, and we consider it a meaningful step to first study the already nontrivial problem of strategic signaling. (2) In modern corporations, decisions of pricing and promotion/marketing are often made by different departments; advertising and pricing may be decoupled decisions. (For similar reasons, some recent works on auctions study "value maximizers" instead of traditional utility maximizers, e.g. Wilkens et al., 2016, 2017; Deng et al., 2024; Balseiro et al., 2024.)

We model competitive advertising as a game of information design (Kamenica and Gentzkow, 2011; Bergemann and Morris, 2019) competing for a sequential searcher. A unit-demand buyer faces N sellers, each seller i with an item of value v_i for the buyer, where v_i is a random variable independently drawn from a known distribution F_i, but is invisible to the buyer, unless she pays an *inspection cost* $c_i > 0$ to observe it. The agent decides the order in which to inspect the sellers, and the time at which to stop searching and purchase from an inspected seller. Each seller aims to maximize the probability that the buyer opens and eventually selects his box. To this end, each sender i, prior to any inspection, commits to a signaling scheme that stochastically maps v_i to a signal. The signal acts as an advertisement. Upon receiving the signal from seller i, the buyer updates her belief about v_i to a posterior. The buyer's optimal search policy is to implement the Index Algorithm on these posteriors. In the jargon of information design, the buyer is the *receiver*, and each seller is a *sender*. We study the game where the N senders compete for being chosen by the receiver, through strategic information revelation.

1.1 Our Contributions

We focus on two algorithmic problems: the sender's problem of best response to a competitive environment, and the equilibrium characterization in symmetric games.

Optimal Information Strategy. Faced with competing sellers, the strategy space of a seller is the set of all possible signaling schemes. It is not only difficult to optimize over this space; given a strategy, it is even challenging to verify whether it is a best response. In Section 2, we use properties of the Index Algorithm to reduce this strategy space to a family of two-dimensional distributions over value v and index θ, and in Section 3 we cast the best response problem as a linear program (LP). Both the primal program and its dual have infinite dimensions, so basic properties in finite-dimensional LPs can no longer be taken for granted. We show that, under mild conditions, both the primal and the dual have well-behaving optimal solutions. We also prove a version of *complementary slackness*. The most technically involved contribution in this part is a procedure that decides whether a given feasible primal solution is optimal, via a construction of dual variables. In most persuasion problems, algorithmic construction of matching dual variables for optimal primal solutions is not known (Dworczak and Martini, 2019). Our construction is made possible by additional spatial relationships between the dual variables in the aforementioned two-dimensional space. The dual variables are determined, via complementary slackness, by reductions to special points and various types of convergent sequences in the primal solution's support set. We illustrate an application of this procedure in Section 5, where we verify that certain strategies are best responses to each other and therefore form an equilibrium.

Existence of Equilibrium. In Section 4, we establish the existence of an equilibrium among the senders when there is no point mass in their priors. We study discrete approximations to the original game and pass the analysis to the limit. In the process, we take special care to rule out possible discontinuity in the senders' utilities caused by ties in the buyer's sequential search.

Symmetric Equilibria. In Section 5, we study symmetric games where the senders share a common convex prior and an identical inspection cost c. We find that the inspection cost c significantly influences the structure of symmetric equilibria.

When c is smaller than all the values in the prior's support, we show that every sender truthfully revealing all values is a unique symmetric equilibrium. This equilibrium maximizes the market efficiency. It suggests that competitive advertising tends to eliminate search friction when the friction is small.

For larger c, we show that full revelation cannot be an equilibrium. Instead, for two-sender games we construct symmetric equilibria where the senders' strategies exhibit a *hinge-convex* structure. In this equilibrium, each sender's values are divided into three parts: low values are essentially discarded, never to be inspected by the buyer; high values are fully revealed; values in the middle are pooled into a range of signals. The strategy makes economic sense: a brand lets a consumer know when there is a good match between its product's features and the consumer's preference, does not waste resource when the match is clearly poor, and otherwise obfuscates the quality of match, taking advantage when the competitors' matches are not good either. In the proof, we guess at the structure of the equilibrium strategy before solving a system of differential equations that govern the structure of such signaling. Properties of the solution confirm that our guess is correct. We then construct duals, using techniques developed in Section 3, to verify that the strategy indeed forms an equilibrium. For uniform priors, we explicitly work out these equilibria and illustrate the change of their structure as c varies in the appendix.

It is worthwhile comparing our findings with the optimal advertising obtained by Anderson and Renault (2006) for a monopoly. A major step in Anderson and Renault is the finding that a monopoly cannot do better than tell a buyer whether or not her value is above a threshold. This is generally not good enough in presence of rivaling sellers, where competition presses the sellers to reveal more information for valuable customers.

1.2 Related Work

The Pandora Box problem (Weitzman, 1978) has seen a fast growing body of algorithmic work on its various extensions (e.g. Doval, 2018; Beyhaghi and Kleinberg, 2019; Boodaghians et al., 2020; Chawla et al., 2020; Fu et al., 2023; Beyhaghi and Cai, 2023a; Hajiaghayi et al., 2025; Banihashem et al., 2025). We refer to Beyhaghi and Cai (2023b) for a survey. The Index Algorithm has served as a model of user search for many works that study markets with competing sellers. See, e.g., Armstrong (2017) for a survey and Derakhshan et al. (2022) and Friedler et al. (2025) for some recent examples.

We model the senders' information revelation as an instance of Bayesian persuasion (Kamenica and Gentzkow, 2011). This framework has inspired a large body of works on information design. See, e.g., Bergemann and Morris (2019); Dughmi (2017); Kamenica (2019) for some recent surveys. In particular, competitive information design examines how multiple senders strategically influence a receiver's decisions through conflicting information. Research in this field falls into two strands: (1) Each sender only reveals information about his own state (Boleslavsky and Cotton, 2015, 2018; Jain and Whitmeyer, 2019; Au and Kawai, 2020, 2021; Sapiro-Gheiler, 2024); and (2) All senders independently disclose information about a common state (Gentzkow and Kamenica, 2016, 2017; Hossain et al., 2024; Ravindran and Cui, 2020). Our work relates to the first strand — each sender i's signal only contains information on v_i. In this strand, Au and Kawai (2020); Boleslavsky and Cotton (2018); Jain and Whitmeyer (2019) study a binary-state setting with two competitive senders. Au and Kawai (2021); Hwang et al. (2019) examine symmetric environments with multi-valued and continuous states. Du et al. (2024) extend the framework to asymmetric priors, and provide a complete characterization of the general equilibria.

Several works study competition in sequential settings (Li and Norman, 2021; Armstrong and Zhou, 2022; He and Li, 2023; Au and Whitmeyer, 2023). Among these, two recent works, Ding et al. (2023) and Hwang and Hwang (2025), consider problems highly related to ours but with crucial differences. In their models, a sender's signal is observed by the buyer *after* inspection, and the buyer sees only the signal (instead of her value). Therefore, compared with the scenario where no signaling is present, the signal in these previous works serves to *obfuscate* the value, whereas in our model it disambiguates the value. In Section 6 we elaborate the difference between the two settings, and compare the receiver's and the senders' utilities in these two models. En route, we give a simplified proof of a result of Ding et al. (2023).

2 Preliminaries

2.1 The Pandora Box Problem

Weitzman (1978) formulated a consumer's sequential search problem as the celebrated *Pandora Box problem*, which we adopt in our model. There are $N \geq 2$ locked boxes and a risk-neutral agent. Each box i has an invisible value v_i independently drawn from a commonly known prior $F_i \in \Delta([0,1])$ (with density denoted by f_i); the box also has an inspection cost $c_i \geq 0$.

Assumption 1. *For each i, F_i is continuous and strictly increasing over $[0,1]$, with $F_i(0) = 0$.*

The agent performs sequential moves: At any time step, she may (a) open a locked box at cost c_i and observe its value v_i, (b) take an opened box and quit, or (c) quit without choosing anything. Her utility is the value of the selected box minus the sum of the inspection costs incurred. Weitzman showed that the following Index Algorithm maximizes the agent's expected utility.

Definition 1 (Index θ_i). *Given prior F_i and cost c_i, the* index θ_i *of box i is the unique solution to the equation $\mathbb{E}_{v_i \sim F_i}[(v_i - \theta_i)_+] = c_i$, where $(x)_+$ denotes* $\max\{x, 0\}$.

Definition 2 (Amortized Value κ_i). *For box i with index θ_i and value v_i, its* amortized value *of box i is $\kappa_i \triangleq \min\{v_i, \theta_i\}$.*

Definition 3 (The Index Algorithm in Weitzman, 1978)**.** *Discard all boxes with negative indices; sort the rest in decreasing order of their indices. Inspect the boxes in this order until the largest observed value exceeds the indices of all remaining locked boxes, or all boxes have been inspected. Take the box with the highest value among those that have been inspected.*

The following lemma, due to Kleinberg et al. (2016), follows from their simplified proof of the Index Algorithm's optimality.

Lemma 1 (Kleinberg et al., 2016)**.** *When the Index Algorithm is implemented, box i is chosen only if $\kappa_i \geq 0$ and $\kappa_i \geq \kappa_j$ for each box $j \neq i$. Moreover, the agent's expected utility is $\mathbb{E}[\max_i(\kappa_i)_+]$.*

2.2 Competitive Information Design

We associate with each box a strategic sender who designs a signaling scheme to disclose information on the box's value. Before any inspection, the searching agent receives all the signals and updates her posterior beliefs for all the values, using the Bayes rule. Thus, the agent acts as a receiver.[1] The senders engage in a game, each trying to maximize the probability of being taken by the receiver.

Timing of the game:

1. Each sender i simultaneously commits to a signaling scheme $(\pi_i(\cdot|\cdot), \mathcal{S}_i)$, where $\mathcal{S}_i$ is the signal space and $\pi_i(\cdot|v_i)$ specifies the conditional probability of sending $s_i \in \mathcal{S}_i$ when v_i is realized.
2. All senders' values are realized independently according to their respective priors, and each sender i sends to the agent a signal $s_i \in \mathcal{S}_i$ according to the conditional distribution $\pi_i(\cdot|v_i)$.
3. The receiver updates her posterior beliefs for all senders and implements the Index Algorithm.

The Index Algorithm remains optimal for the receiver. [2] When running the Index Algorithm, the receiver calculates the indices using the posterior distributions.

Sender's actions as joint distributions. The action space of each sender is the set of all valid signaling schemes. Each signal induces a posterior distribution, and each posterior distribution uniquely corresponds to an index according to Definition 1. As Lemma 1 shows, both parties' utilities are determined by the index of the posterior and the realized value. If two signals induce posteriors with the same index, they can be merged into one equivalently. Therefore, it is without loss of generality to think that each signal corresponds to a distinct index. A feasible action of the sender can then be seen as the following joint distribution on the value and the index.

Definition 4 (Feasible Joint Distribution)**.** *For each sender i, given prior F_i and cost c_i, any valid signaling scheme gives rise to a* feasible joint distribution *$G_i(v, \theta)$ (with density $g_i(v, \theta)$) satisfying*

$$\mathbb{E}_{v \sim G_i(\ \cdot\ |\ \theta)}[(v - \theta)_+] = c_i\ , \quad \forall \theta \in [-c_i, 1 - c_i]\ , \tag{1}$$

$$\int_{\theta=-c_i}^{1-c_i} g_i(v, \theta)\ \mathrm{d}\theta = f_i(v)\ , \quad \forall v \in [0, 1]\ , \tag{2}$$

[1] In this work, "box" and "sender" are used interchangeably, as are "agent" and "receiver".

[2] We assume the receiver is myopic and unable to commit to sub-optimal search strategies in order to influence the sellers' signaling strategies. This is typical in the information design literature, and is appropriate for our economic setting, where sellers are typically long-term retailers whereas a buyer neither reenters the market after a purchase nor strategizes with the other buyers.

where $G_i(\,\cdot\,|\,\theta)$ denotes the posterior distribution corresponding to index θ. The set of such feasible joint distributions is denoted as $\mathcal{G}(F_i, c_i) \subseteq \Delta([0,1] \times [-c_i, 1-c_i])$.

Lemma 2. *The action space $\mathcal{G}(F_i, c_i)$ is convex and compact.*

As a shorthand, we refer to such a feasible joint distribution as a *2-D distribution*. Given cost c and a 2-D distribution G, let $G_\theta(\cdot)$ be its marginal distribution of index: $G_\theta(x) \triangleq G(1, x)$ for any $x \in [-c, 1-c]$. Let $G_{\cdot|\theta}(\cdot)$ be the conditional distribution of value given one index θ: [3]

$$G_{\cdot|\theta}(x) \triangleq \frac{\int_{t=0}^{x} \, g(t,\theta) \, \mathrm{d}t}{\int_{t=0}^{1} \, g(t,\theta) \, \mathrm{d}t} \ , \quad \forall x \in [0,1] \ .$$

Timing of the game with 2-D distributions: The original game is equivalent to the following game with 2-D distributions as the senders' actions:

1. Each sender i chooses a 2-D distribution $G_i \in \mathcal{G}(F_i, c_i)$ subject to Constraints (1) and (2).
2. The receiver realizes for each sender i a tuple $(v_i, \theta_i) \sim G_i$, and chooses the sender $j \in \mathrm{argmax}_{i:\kappa_i \geq 0} \kappa_i$, where the amortized value $\kappa_i = \min\{v_i, \theta_i\}$. When indifferent between multiple senders, the receiver chooses one uniformly at random. If $\kappa_i < 0$ for all i, the receiver quits without choosing anything.

Senders' utilities. Each sender i gains utility 1 if chosen by the receiver and 0 otherwise. Given strategy profile $(G_1, ..., G_N)$ and sender i's realized tuple $(v_i, \theta_i) \sim G_i$, we denote sender i's interim utility as $u_i(\kappa_i, G_{-i})$, where $\kappa_i = \min\{v_i, \theta_i\}$, and $G_{-i} \triangleq (G_1, ..., G_{i-1}, G_{i+1}, ..., G_N)$ denotes the other senders' strategies. We denote sender i's *amortized value distribution* by $K_i(\kappa) \triangleq G_i(1, \kappa) + G_i(\kappa, 1-c_i) - G_i(\kappa, \kappa)$ for $\kappa \in [-c_i, 1-c_i]$. By Lemma 1, if $\kappa_i < 0$, then $u_i(\kappa_i, G_{-i}) \triangleq 0$; otherwise,

$$u_i(\kappa_i, G_{-i}) \triangleq \prod_{j \neq i} \mathbf{Pr}\left[\kappa_j < \kappa_i\right] = \prod_{j \neq i} \left[G_j(1, \kappa_i) + G_j(\kappa_i, 1-c_j) - G_j(\kappa_i, \kappa_i)\right] = \prod_{j \neq i} K_j(\kappa_i) \ . \tag{3}$$

The expression in (3) ignores ties. In most scenarios we consider, there cannot be a probability mass in any strategy G_i at equilibrium, so this is without loss of generality. Given a strategy profile $(G_1, \ldots, G_N)$, we denote sender i's expected utility as $U_i(G_i, G_{-i}) \triangleq \mathbb{E}_{(v_i,\theta_i) \sim G_i}[u_i(\min\{v_i, \theta_i\}, G_{-i})]$.

Solution concept. We study *Nash equilibrium* (hereafter equilibrium) in the game among the senders. A strategy G_i^* of sender i is a *best response* to the other senders' strategies G_{-i}^* if $G_i^* \in \mathrm{argmax}_{G_i \in \mathcal{G}(F_i, c_i)} U_i(G_i, G_{-i}^*)$. A strategy profile $G^* = (G_1^*, \ldots, G_N^*)$ forms an equilibrium if G_i^* is a best response to G_{-i}^* for each sender $i \in [N]$. Since the strategy space $\mathcal{G}(F_i, c_i)$ is convex and compact (Lemma 2), there is no difference between pure equilibria and mixed equilibria.

3 Optimal Information Strategy

In this section, we analyze the sender's problem of best responding to the competitors' strategies. We show the existence of optimal solutions for both the primal and dual problems (Theorem 1), and

[3] Hereafter, all distributions are represented by their cumulated density functions (CDFs), unless specified. Besides, we use $\mathrm{supp}(G)$ to represent the support set of a CDF G.

derive a version of complementary slackness conditions (Theorem 2), which can be used to verify the optimality of a pair of feasible solutions. Lastly, we give a method to decide if a given primal solution is optimal (Theorem 3); this is by constructing a dual solution (Algorithms 1 and 2).

We write the sender's problem as an infinite-dimensional LP.[4] For simplicity, denote the interim utility of (v, θ) by $p(v, \theta) \triangleq u(\min\{v, \theta\})$. Let $q(v, \theta) \triangleq (v - \theta)_+ - c$. Let the density g of a 2-D distribution be variables of the program. A sender's best response is an optimal solution to:

$$\max_{g} \quad \int_{v=0}^{1} \int_{\theta=-c}^{1-c} p(v,\theta) g(v,\theta) \, \mathrm{d}\theta \, \mathrm{d}v \tag{$\mathcal{P}_{\mathsf{BR}}$}$$

$$\text{subject to} \quad \int_{\theta=-c}^{1-c} g(v,\theta) \, \mathrm{d}\theta = f(v) \ , \quad \forall v \in [0,1] \tag{4}$$

$$\int_{v=0}^{1} q(v,\theta) g(v,\theta) \, \mathrm{d}v = 0 \ , \quad \forall \theta \in [-c, 1-c] \tag{5}$$

Constraint (4) ensures that the marginal of the 2-D distribution matches the prior F. Constraint (5) ensures that the conditional distribution of v given any θ does yield the index θ.

We next consider the dual problem. Let the dual variables for Constraint (4) be $\lambda(v)$ for each $v \in [0, 1]$, and $\mu(\theta)$ for Constraint (5) for each $\theta \in [-c, 1-c]$. Then the dual problem is:

$$\min_{\lambda, \ \mu} \quad \int_{0}^{1} \lambda(v) f(v) \, \mathrm{d}v \tag{$\mathcal{D}_{\mathsf{BR}}$}$$

$$\text{subject to} \quad \lambda(v) + \mu(\theta) q(v,\theta) \geq p(v,\theta) \ , \quad \forall (v,\theta) \in [0,1] \times [-c, 1-c] \tag{6}$$

For continuous u, the primal must have an optimal solution by the convexity and compactness of $\mathcal{G}(F, c)$ (Lemma 2). We show that the dual also has an optimal solution, with additional properties.

Theorem 1 (Existence of an Optimal Solution to $\mathcal{D}_{\mathsf{BR}}$)**.** *If u is L-Lipschitz continuous over $[-c, 1-c]$ for some $L > 0$,[5] the dual problem has an optimal solution (λ^*, μ^*) such that: (i) $\mu^*(\theta) \in [-L, 0]$ for each $\theta \in [-c, 1-c]$; and (ii) λ^* is non-decreasing and continuous over $[0, 1]$.*

Complementary slackness also holds, and can be used to verify whether a given pair of feasible solutions is optimal both to the primal and dual problems.

Theorem 2 (Complementary Slackness)**.** *Given feasible primal solution g^* and feasible dual solution (λ^*, μ^*), g^* and (λ^*, μ^*) are both optimal if and only if*

$$\int_{v=0}^{1} \int_{\theta=-c}^{1-c} [\lambda^*(v) + \mu^*(\theta) q(v,\theta) - p(v,\theta)] g^*(v,\theta) \, \mathrm{d}\theta \, \mathrm{d}v = 0 \ . \tag{7}$$

Besides, strong duality holds. That is, (7) implies $\int_{v=0}^{1} \int_{\theta=-c}^{1-c} p(v,\theta) g^*(v,\theta) \, \mathrm{d}\theta \, \mathrm{d}v = \int_0^1 \lambda^*(v) f(v) \, \mathrm{d}v$.

In light of this condition, we can interpret the optimal dual variables intuitively. For any

[4] The subscript i is omitted in this section, as we focus on a fixed sender. With slight abuse of notation, we denote the sender's interim utility function by $u(x) \triangleq u(x, G_{-i})$ for $\forall x \in [-c, 1-c]$, given the other senders' strategies G_{-i}.

[5] In Euclidean space, a function $f : S \to \mathbb{R}$ is Lipschitz continuous if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L \cdot |x - y|$ for all $x, y \in S$.

value-index pair $(v,\theta) \in \mathrm{supp}(G^*)$ with $g^*(v,\theta) > 0$, (6) and (7) imply that

$$\lambda^*(v) = \max_{\theta' \in [-c,1-c]} p(v,\theta') - \mu^*(\theta')q(v,\theta') = p(v,\theta) - \mu^*(\theta)q(v,\theta) \ ,$$

where $p(v,\theta)$ is the utility brought by (v,θ), and $-\mu^*(\theta)q(v,\theta)$ captures its spillover effect on other values sharing the same index θ. Thus, intuitively speaking, $p(v,\theta) - \mu^*(\theta)q(v,\theta)$ can be seen as the amortized utility of the pair (v,θ), and $\lambda^*(v)$ represents the utility contribution of value v. In any best response, each value v select indices θ that maximize the amortized utility as its contribution.

Next, we provide a method to verify whether a given primal solution is optimal. Before this, the following necessary condition for optimality quickly rules out many strategies. The intuition is that any amortized value $\kappa < 0$ contributes zero utility, according to Lemma 1.

Definition 5 (Threshold Value)**.** *For any 2-D distribution $G \in \mathcal{G}(F,c)$, define the set $T \triangleq \{v \in [0,c] \mid \mathrm{supp}(G_{\cdot|\theta}) \cap [0,v) = \emptyset \ \forall \theta \in (v-c, 1-c],$ and $\mathrm{supp}(G_{\cdot|\theta}) \cap (v,1] = \emptyset \ \forall \theta \in [-c, v-c)\}$. The threshold value $\underline{v}$ of G is $\sup T$ if T is non-empty, and 0 otherwise.*

Lemma 3. *Any best response G^* has a threshold value $\underline{v} > 0$.*

In any best response, values from $[0,c]$ are divided into two parts by $\underline{v}$: those too small to contribute utility have negative indices, while the others, pooled with larger values, form positive indices and generate positive utility. See the left panel of Figure 1 for clarification.

Guided by the complementary slackness conditions, we propose Algorithms 1 and 2 to construct the dual variables λ and μ respectively (Algorithm 2 is in Appendix B). With this construction, the following theorem serves as a verification tool for the optimality of the primal problem.

Theorem 3 (Best Response Verification)**.** *Assume that: (i) u is differentiable and Lipschitz continuous everywhere; and (ii) for every $(v,\theta) \in \mathrm{supp}(G)$, any open neighborhood of (v,θ) has strictly positive probability under G. If one feasible solution (λ,μ) can be constructed according to Algorithms 1 and 2, then G and (λ,μ) are both optimal to the primal and dual problems.*

Construction Method. We proceed with the construction in two steps. We first determine $\lambda(v)$ for each $v \in [0,1]$ using spatial relationships between the dual variables (Algorithm 1), and then check whether there is a set of $\mu(\theta)$ for each $\theta \in [-c, 1-c]$ that constitutes a feasible dual solution along with the λ constructed (Algorithm 2). The logic of Algorithm 2 is relatively straightforward, so we focus on explaining the construction of λ in the main text.

To show the details of Algorithm 1, we first introduce key definitions: A sequence $\{(v^m,\theta^m)\}_{m\in\mathbb{Z}^+}$ is called a *monotone G-sequence* if: (i) $(v^m,\theta^m) \in \mathrm{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; and (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone. Furthermore, if the sequence satisfies condition (iii) with *strict* monotonicity, then we call it a *strictly monotone G-sequence.*

Regarding the construction of λ, we base our approach on characterizing the necessary conditions for both G and (λ,μ) to be optimal simultaneously. We provide graph illustrations for Algorithm 1 in the right panel of Figure 1. Recall that, due to the complementary slackness conditions, $\lambda(v)$ represents the contribution of value v to the utility, given G and (λ,μ) are both optimal. For any given value $v \in [0,1]$, the determination of $\lambda(v)$ falls into exact one of five cases. Among these, there are two cases where we can directly determine the corresponding $\lambda(v)$:

- **Case 1:** If $v \in [0,\underline{v}]$, then Algorithm 1 returns $\lambda(v) = 0$. Based on the intuitions of the

Algorithm 1 LAMBDA (v): Construction of the dual variable λ.

Input: Strategy G; interim utility u; cost c; threshold value $\underline{v}$; and value $v \in [0, 1]$.
Output: The corresponding $\lambda(v)$.

if $v \in [0, \underline{v}]$ **then**
    $\lambda(v) \leftarrow 0$ and **return** $\lambda(v)$
if $(v, v - c) \in \text{supp}(G)$ **then**
    $\lambda(v) \leftarrow u(v - c)$ and **return** $\lambda(v)$
if there is a monotone G-sequence converging to $(v, v - c)$ **then**
    $\lambda(v) \leftarrow u(v - c)$ and **return** $\lambda(v)$
else if there is a strictly monotone G-sequence converging to (v, θ_1) with $\theta_1 \neq v - c$ **then**
    $\lambda(v) \leftarrow u'(\theta_1)q(v, \theta_1) + p(v, \theta_1)$ and **return** $\lambda(v)$
else
    There is a $\theta_2 \neq v - c$ such that $(v - \epsilon, v) \subseteq \text{supp}(G_{\cdot|\theta_2})$ for some $\epsilon > 0$.
    $\underline{v}_{\theta_2} \leftarrow \inf \text{supp}(G_{\cdot|\theta_2})$
    $\lambda(v) \leftarrow \frac{\text{LAMBDA}(\underline{v}_{\theta_2}) - p(\underline{v}_{\theta_2}, \theta)}{q(\underline{v}_{\theta_2}, \theta)} q(v, \theta) + p(v, \theta)$ and **return** $\lambda(v)$

threshold value $\underline{v}$ mentioned above, all values below $\underline{v}$ contribute no positive utility, thus their contribution $\lambda(v)$ must be all zero.

- **Case 2:** If $(v, v - c) \in \text{supp}(G)$, then Algorithm 1 returns $\lambda(v) = u(v - c)$. In this case, $q(v, v - c)$ is equal to zero, thereby eliminating the effect of $-\mu(v - c)$. The point $(v, v - c)$ does not affect any other values sharing the same index, thus its amortized contribution is precisely the utility $u(v - c)$ itself, and we can directly obtain $\lambda(v) = u(v - c)$.

If the above two simple cases cannot determine $\lambda(v)$, since we assumed the prior distribution has a positive density everywhere on $[0, 1]$, we conclude there must exist a monotone G-sequence converging to (v, θ) for some $\theta \in [0, 1 - c]$, and this sequence must necessarily be dense. Based on the convergence properties of this sequence, we then classify the analysis into three cases:

- **Case 3.1:** Assume there exists a monotone G-sequence $\{(v^m, \theta^m)\}_{m \in \mathbb{Z}^+}$ converging to $(v, v - c)$. Since the continuity of λ and the boundedness of μ, whether $(v, v - c)$ belongs to $\text{supp}(G)$ or not, it both holds that $\lambda(v) = \lim_{m \to \infty} -\mu(\theta^m)q(v^m, \theta^m) + p(v^m, \theta^m) = u(v - c)$.

- **Case 3.2:** Assume there exists a strictly monotone G-sequence $\{(v^m, \theta^m)\}_{m \in \mathbb{Z}^+}$ converging to (v, θ_1) for some $\theta_1 \neq v - c$. Theorem 2 implies that $\lambda(v^m) = -\mu(\theta^m)q(v^m, \theta^m) + p(v^m, \theta^m)$ for any $m \in \mathbb{Z}^+$. During the convergence of the monotone G-sequence, we can obtain infinitely many such equalities in the neighborhood of (v, θ_1). The simultaneous satisfaction of these infinite equalities, combined with the specific formula of $-\mu(\theta^m)q(v^m, \theta^m) + p(v^m, \theta^m)$, allows us to conclude that $\lambda(v) = u'(\theta)q(v, \theta_1) + p(v, \theta_1)$.

- **Case 3.3:** If none of the above conditions hold, since the prior has positive density everywhere, there exists $\theta_2 \neq v - c$ such that $(v - \epsilon, v) \subseteq \text{supp}(G_{\cdot|\theta_2})$ for some $\epsilon > 0$. In this case, by the continuity of the function λ, we can reduce the task of solving $\lambda(v)$ to solving $\lambda(\inf \text{supp}(G_{\cdot|\theta_2}))$. We recall Algorithm 1 with parameter $\inf \text{supp}(G_{\cdot|\theta_2})$ to solve this case.

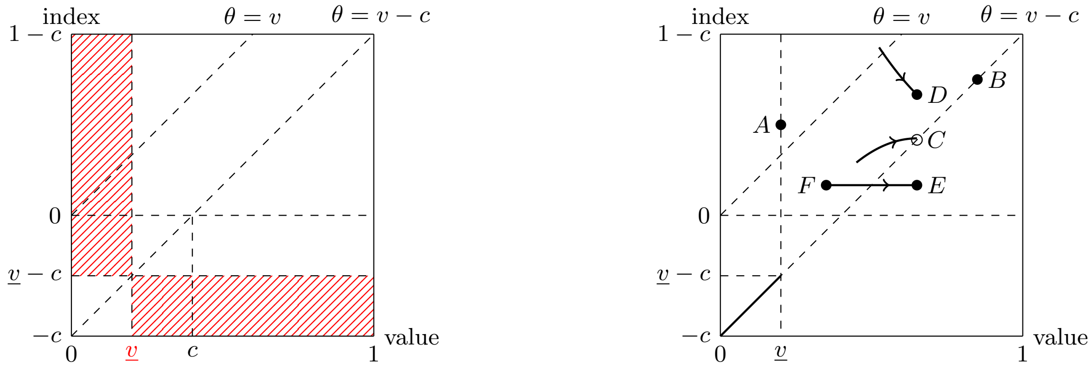

Figure 1: The left panel clarifies the threshold value $\underline{v}$. The support set of any best response 2-D distribution has no intersection with the red shaded regions. The right panel illustrates Algorithm 1. The solid dots and black curves represent the support set of one best response candidate G. The hollow dots represent the limit points of some monotone G-sequence. The arrows indicate the directions of convergence. All five cases above are shown in this figure. Case 1: Let $\lambda(v_A) = 0$ since $v_A = \underline{v}$; Case 2: Let $\lambda(v_B) = u(v_B - c)$ since $\theta_B = v_B - c$; Case 3.1: Let $\lambda(v_C) = u(v_C - c)$ since there is a monotone G-sequence converging to (v_C, θ_C); Case 3.2: Let $\lambda(v_D) = u'(\theta_D)q(v_D, \theta_D) + p(v_D, \theta_D)$ since there is a strictly monotone G-sequence converging to (v_D, θ_D); Case 3.3: Let $\lambda(v_E) = \frac{\lambda(v_F) - p(v_F, \theta_F)}{q(v_F, \theta^F)} q(v_E, \theta_E) + p(v_E, \theta_E)$ since $(v_F, \theta_E) \subseteq \mathrm{supp}(G_{\cdot|\theta_E})$.

4 Equilibrium Existence

In this section, we establish the existence of an equilibrium under mild assumptions.

Theorem 4 (Equilibrium Existence)**.** *There exists an equilibrium in the game among the senders if each sender i's prior distribution F_i is continuous over* $[0, 1]$ *and* $F_i(0) = 0$.

The theorem holds whenever the tie-breaking rule is not degenerate, in the sense that, conditioning on a tie, no sender wins with probability 1. The continuity of the prior distribution is crucial for the theorem to hold. We provide an example in Appendix C to show that an equilibrium may fail to exist when there is probability mass in the priors.

Two challenges stand out in the proof: the sender's action space is potentially infinite, with each action represented by an infinite-dimensional 2-D distribution; and the sender's utility may be discontinuous due to the possible ties. To handle these, we analyze a series of *discrete approximations*, with increasingly finer granularity, of the game and pass the analysis to the limit. First, we construct a series of finite games by discretizing the value support and finalizing the linear constraints. Then, in each finite game, we are able to leverage Nash's Theorem to directly show the existence of a discrete equilibrium. We finally show that the sequence of discrete equilibria has a limit point that constitutes an equilibrium in the original game. See Appendix C for details.

5 Symmetric Equilibrium

In this section, we consider symmetric games where all senders share a common prior F and a search cost $c > 0$. For convex priors F (with non-decreasing density f), we show that if c is smaller than the lowest value in the support, the unique symmetric equilibrium has all senders fully reveal their values (Theorem 5); for larger c, with two senders, we construct a symmetric equilibrium where the senders fully reveal their high values but pool values in the middle to attract some buyers with values lower than c (Theorem 6).

5.1 Convex Prior and Low Cost

Theorem 5. *If all senders share the same prior F that is (weakly) convex, with all values in its support at least c, then each sender fully revealing his value is the unique symmetric equilibrium.*

For an intuition that full revelation constitutes an equilibrium, observe that, for each sender, the interim utility $u(\kappa) = F^{N-1}(\kappa + c)$ is convex in this scenario; any pooling of values decreases the expected utility by Jensen's inequality. Intuitively, fully revealing the values is a best response.

The uniqueness of this symmetric equilibrium is more nontrivial. As a starting point, note that the senders' game is constant-sum for such a small c: since all values are at least c, the index of each sender is always non-negative, regardless of the signaling; therefore, the buyer never quits without selecting a seller. Thus, in any symmetric equilibrium, each sender's utility must be $1/N$. We show that if all senders except i use a common strategy which does not always fully reveal the values, sender i can get a utility strictly higher than $1/N$ by fully revealing v_i. The uniqueness of the symmetric equilibrium follows.

Note that the market achieves maximum efficiency under such an equilibrium.

5.2 Convex Prior and High Cost

We now turn to games with a common convex prior F but a higher search cost $c > \inf \operatorname{supp}(F)$. Values smaller than c, if fully revealed, will never be inspected or generate utility; the senders are incentivized to pool some of them with higher values to induce inspection. The symmetric equilibrium we construct for two-sender games illustrates this, and is substantially more involved than the equilibrium in Theorem 5.

It is instructive to see why full revelation no longer constitutes an equilibrium. If both senders fully reveal their values, each sender's interim utility $u(\kappa)$ changes from 0 (for $\kappa < 0$) abruptly to $F(c)$ at $\kappa = 0$. This non-convexity makes pooling profitable: e.g., by pooling values $c - \epsilon$ and $c + 2\epsilon$ in equal proportions to obtain amortized value ϵ, the sender gains a proportion of utility $2u(\epsilon) - u(2\epsilon) = 2F(c+\epsilon) - F(c+2\epsilon)$. As $F(c+\epsilon) > F(c) > 0$ and $\lim_{\epsilon \to 0} F(c+\epsilon) - F(c+2\epsilon) = 0$, for sufficiently small ϵ, this is strictly positive utility gain.

This suggests that, in a symmetric equilibrium, some values smaller than c should be pooled with higher values to produce positive amortized values. What should be the shape of u in this region? By Jensen's inequality, the sender tends to reveal more where u is strictly convex; conversely, he tends to pool more (and reveal less) where u is strictly concave. For partial pooling to sustain in a two-sender equilibrium, it is conceivable that u should be linear in the pooled region. Theorem 6 constructs a symmetric equilibrium based on this intuition. It is remarkable that this rough intuition transpires without further modification: the only pooling in the strategy serves to levitate the values just below c, and the strategies verifiably constitute an equilibrium.

Theorem 6 (Hinge-convex Signaling)**.** *For two senders with a common convex prior F and search cost c larger than the lowest value in the support of F, there is a symmetric equilibrium where each sender's amortized value distribution $K(\kappa)$ is of the following hinge-convex structure: for a certain $\theta_1 \in (-c, 0)$, $\theta_2 \in (0, 1-c]$, and $\rho > 0$,*

$$K(\kappa) = \begin{cases} F(\kappa + c) & \text{if } \kappa \in [-c, \theta_1] \ , \\ F(\theta_1 + c) & \text{if } \kappa \in (\theta_1, 0] \ , \\ \min\{\rho \cdot \kappa + F(\theta_1 + c), 1\} & \text{if } \kappa \in (0, \theta_2] \ , \\ F(\kappa + c) & \text{if } \kappa \in (\theta_2, 1-c] \ , \end{cases}$$

where $K(\theta_2) = F(\theta_2 + c)$ if $\theta_2 < 1 - c$, and $K(\theta_2) = 1$ if $\theta_2 = 1 - c$; in the latter case the fourth part is degenerate. Moreover, such θ_1, θ_2 and ρ are unique for this type of symmetric equilibrium.

In this equilibrium, values smaller than $\theta_1 + c$ are fully revealed (and produce negative amortized values, with zero utility). Most interestingly, the values between $\theta_1 + c$ and c are pooled with higher values to produce positive amortized values evenly distributed on an interval starting from 0. In the non-degenerate case, not all values above $\theta_1 + c$ are pooled (i.e., $\theta_2 < 1 - c$), in which case the interval of pooled amortized values ends at θ_2, and values larger than $\theta_2 + c$ are again fully revealed (Figure 2). In the degenerate case, all the values above $\theta_1 + c$ are pooled (i.e., $\theta_2 = 1 - c$), and the amortized value peaks somewhere less than θ_2 (Figure 3).

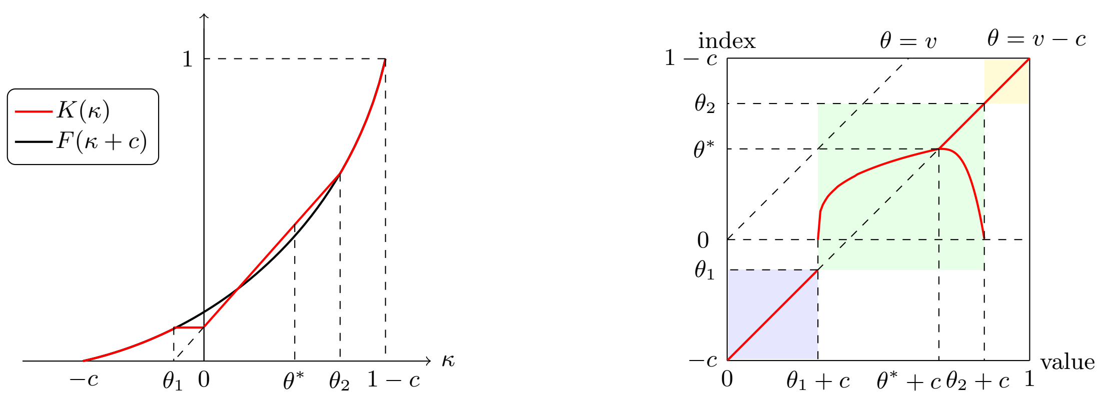

Figure 2: An example of Theorem 6. In the left panel, the red curve represents the amortized value distribution $K(\kappa)$, and the black curve the shifted prior $F(\kappa + c)$. In the right panel, the red curves represent the support of the 2-D distribution in the equilibrium. In the light blue and light yellow regions, low values and high values are fully revealed; in the light green region, values are pooled to produce amortized values uniformly distributed on an interval. In particular, values above $\theta^* + c$ and those below are paired in the pooling, where $\theta^* \triangleq f^{-1}(\rho)$. In this non-degenerate case, $\theta_1' = \theta_2$.

Proof Sketch: We start with an important fact: the amortized value distribution K of any 2-D distribution is a mean-preserving contraction (MPC) of the shifted prior. [6]

Lemma 4. *$\int_{-c}^{t-c} K(x)\, \mathrm{d}x \le \int_0^t F(x)\, \mathrm{d}x$ for all $t \in [0, 1]$ and $\int_{-c}^{1-c} K(x)\, \mathrm{d}x = \int_0^1 F(x)\, \mathrm{d}x$.*

[6] Recall that a distribution G is an MPC of a distribution F if F *majorizes* G, i.e., $\int_{-\infty}^{t} F(x)\, \mathrm{d}x \ge \int_{-\infty}^{t} G(x)\, \mathrm{d}x$ for any $t \in \mathbb{R}$ with equality at $t = +\infty$. The celebrated Blackwell Theorem states that, G forms an MPC of F if and only if there is a signaling scheme on F whose posterior mean has distribution G. The theorem is not directly applicable in our setting, as the amortized value results from the calculation of the index.

To implement the intuition outlined before the theorem statement, we first use Lemma 4 to determine θ_1 and θ_2, and then show that the amortized value distribution given in the theorem can indeed be realized by a 2-D distribution. Finally, we verify that the resulting strategy forms a symmetric equilibrium.

Step 1: Determining θ_1 and θ_2. We would like to pool values in $(\theta_1 + c, \theta_2 + c]$ so that the resulting amortized value is distributed uniformly on an interval $[0, \theta_1']$ for some $\theta_1' \leq \theta_2$; i.e., $K(\kappa)$ is a line segment with slope ρ on $(\theta_1, \theta_1']$. (As will be clear, in non-degenerate cases, θ_1' is simply θ_2.) We also would like $K(\kappa)$ to match $F(\kappa + c)$ for $\kappa \notin (\theta_1, \theta_1']$. By Lemma 4, on the two endpoints we must have $K(\theta_1) = F(\theta_1 + c)$ and $K(\theta_2) = F(\theta_2 + c)$. Therefore, $\rho \cdot \theta_1' = F(\theta_1' + c) - F(\theta_1 + c)$. (See Figure 2.) Another identity follows from considering the boundary value $\theta_1 + c$: at equilibrium, this value should be indifferent between being pooled with some higher value to produce amortized value 0, yielding utility $F(\theta_1 + c)$, and being truthfully revealed to produce amortized value $\theta_1 < 0$, yielding utility 0. Hence, the utility gain from pooling, which is $F(\theta_1 + c)$, should be equal to the loss from pooling, which comes from lowering the amortized value of some higher value(s) by the amount of $|\theta_1|$; since K is linear throughout the pooled region, this reduction in the amortized value causes a loss of utility $\rho|\theta_1|$. Therefore, $F(\theta_1 + c) = \rho|\theta_1|$. In the appendix, we show that there exist unique θ_1 and θ_1' that solve

$$\frac{F(\theta_1 + c)}{|\theta_1|} = \frac{F(\theta_1' + c) - F(\theta_1 + c)}{\theta_1'} = \rho \ .$$

If $K(\theta_1') < 1$, then $\theta_2 = \theta_1'$ (as in Figure 2); otherwise, $\theta_2 = 1 - c$, which is the degenerate case (see Figure 3 for an illustration).

Step 2: Construction of the corresponding 2-D distribution. We sketch the proof for the non-degenerate case $\theta_1' = \theta_2 < 1 - c$. The degenerate case (where $\theta_1' < \theta_2 = 1 - c$) is proved analogously. As planned, values in $[0, \theta_1 + c) \cup (\theta_2 + c, 1]$ should be truthfully revealed, making $K(\kappa)$ match the prior $F(\kappa + c)$ for $\kappa \in [-c, \theta_1] \cup [\theta_2, 1 - c]$; values in $[\theta_1 + c, \theta_2 + c]$ should be pooled to produce amortized values uniformly distributed on $[0, \theta_2]$. We divide these values into high and low ones. Since $\int_{\theta_1+c}^{\theta_2+c} f(v)\ \mathrm{d}v = F(\theta_2 + c) - F(\theta_1 + c) = K(\theta_2) - K(\theta_1) = \rho(\theta_2 - \theta_1)$, there exists $\theta^* \in (\theta_1, \theta_2)$ such that $f(v) \geq \rho$ iff $v \geq \theta^* + c$ (recall that the density f is non-decreasing). We say a value is *high* if it is at least $\theta^* + c$, and otherwise it is *low*. Each high value v is to be truthfully revealed with probability $\frac{k}{f(v)}$, and otherwise pooled with a certain low value to form an index. In other words, each high value is paired with a low value in the pooling (see Figure 2 right panel). This pairing is exquisite, since the resulting amortized value must be uniformly distributed, while Constraints (1) and (2) keep being satisfied. We handle this by formulating and solving a system of ordinary differential equations (ODEs). Each index $\theta \in [0, \theta^*]$ is formed by pooling a low value $\alpha(\theta)$ and a high value $\beta(\theta)$: $\alpha(\cdot)$ and $\beta(\cdot)$ are the functions to be solved for in the ODEs. For the initial condition, the lowest value is paired with the highest value to form index 0, i.e., $\alpha(0) = \theta_1 + c$, $\beta(0) = \theta_2 + c$. We are able to explicitly solve the ODEs governing $\alpha(\cdot)$ and $\beta(\cdot)$ and show (i) $\alpha'(\theta) \geq 0$ and $\beta'(\theta) \leq 0$ for any $\theta \in [0, \theta^*]$; and (ii) the process terminates at $\theta = \theta^*$ with $\alpha(\theta^*) = \beta(\theta^*) = \theta^* + c$. Therefore, not only is the hinge-shaped distribution of amortized value implementable, but we also have an explicit construction to implement it.

Step 3: Equilibrium verification. We verify that the hinge-shaped K indeed forms a symmetric equilibrium. This we do via techniques in Section 3. In particular, we explicitly construct duals using Algorithms 1 and 2 and invoke Theorem 2 to certify that the hinge-shaped K is a best response to itself. Details are relegated to the appendix.

Example 1. *For uniform prior $F(v) = v$ for $v \in [0, 1]$, the equilibrium in Theorem 6 is given by $\theta_1 = \frac{c^2-c}{2-c}$, $\theta_2 = 1 - c$, and $\rho = 1 - c$ (see Figure 3). This is a degenerate case, as $\theta_2 = 1 - c$. In Appendix D, we discuss how the equilibrium structure changes as the search cost c varies.*

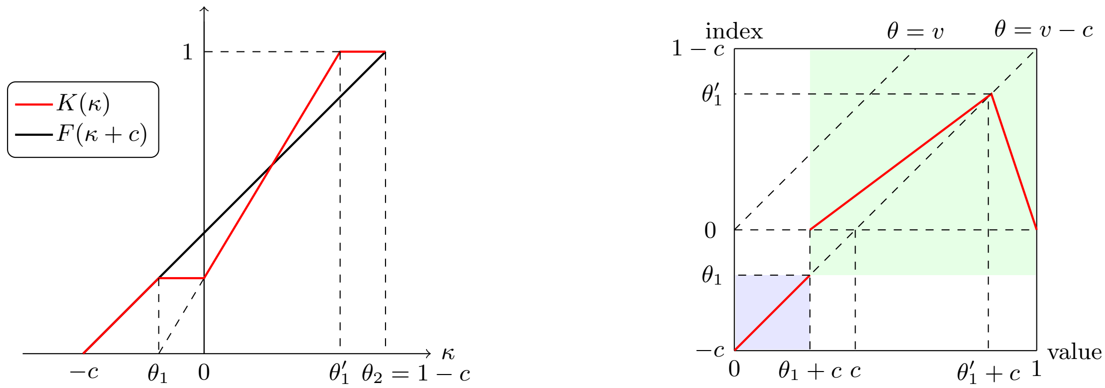

Figure 3: The case of uniform prior. Let $\theta_1' = \frac{2c^2-4c+2}{2-c}$ denote the right endpoint of the linear segment. All elements share the same meanings as in Figure 2. This is a degenerate case where $\theta_1' < \theta_2 = 1-c$, causing the full-revelation interval $[\theta_2 + c, 1]$ (yellow region in Figure 3) to collapse.

6 Comparing Models: Timing of Signaling

As mentioned in Related Works, Ding et al. (2023); Hwang and Hwang (2025) study a closely related model, where the sender's signal is seen by the buyer *after* inspection, and the buyer sees only the signal instead of her value. We call this setting one of *information obfuscation*; in comparison, our setting is one of *information revelation*. In this section, we compare the informational properties of these two settings and then compare the senders' and the receiver's utility.

6.1 Informational Properties

Extreme cases highlight the differences between the two models. In our model, all senders adopting the *full revelation* strategy let the buyer know all the values before searching; her utility would be the first best $\mathbb{E}[\max_i(v_i - c_i)_+]$. If no sender sends any signal, the buyer gets no additional information. She just searches with the original priors $F_1, \cdots, F_N$. In contrast, this original problem corresponds to the case of full revelation in the setting of information obfuscation. In this setting, if the senders send no signals, the distribution of each v_i degenerates to a point mass on $\mathbb{E}[v_i]$, which reduces the buyer's utility to $\max_i(\mathbb{E}[v_i] - c_i)_+$.

The buyer's search behavior also differs in the two models. In the information revelation setting, the buyer's search depends on the *signals* she receives, updating her posteriors based on these signals. In the obfuscation setting, the search depends only on the *signaling schemes* — once the senders commit to the schemes, the value distribution of each seller becomes an MPC of the original prior, and the receiver searches using these MPCs, observing a signal only after inspecting a seller.

6.2 Senders' and Receiver's Utilities

Fixing the priors of values and search costs, let U be the buyer's utility when she implements the Index Algorithm in the original Pandora Box problem.

Proposition 1. *U is weakly lower than the buyer's utility in the information revelation setting, and weakly higher than the buyer's utility in the information obfuscation setting, regardless of the senders' signaling schemes in both settings.*

The second statement was first shown by Ding et al. (2023) (Theorem 3.1); we consider our proof here considerably simpler. Intuitively, the more information accessible to the receiver, the higher her utility should be. The proofs of these two claims both rely on constructing intermediate search strategies that connect each of the two settings.

The next proposition considers a sender choosing between revealing or obfuscating information.

Proposition 2. *Fix all priors and costs, and consider a sender i who unilaterally considers deploying a signaling scheme, we have: (i) when his cost $c_i = 0$, any strategy of obfuscation gives him weakly higher utility than revelation; and (ii) when his cost $c_i > \mathbb{E}_{v_i \sim F_i}[v_i]$, any strategy of revelation gives him weakly higher utility than obfuscation.*

The cost affects the receiver's willingness to search and her optimal search strategy. For a seller i with search cost $c_i = 0$, the receiver will definitely inspect the seller for free. Thus, an information-revealing sender can do nothing, since any revelation only serves to influence the searching order of the receiver. In contrast, when his cost is sufficiently high, an obfuscating sender always has a negative index regardless of the strategy, thus will never be inspected under the Index Algorithm. Besides, in Appendix E, we conjecture that, as the cost c_i increases from 0 to 1, the difference between the sender's utility of optimal revelation and that of optimal obfuscation is monotone non-decreasing.

7 Conclusion

In this work, we study a scenario where sellers use informative advertisements to compete for a buyer who performs sequential searches in face of search friction. Following Anderson and Renault (2006), we model this as a competitive information design problem, with the receiver adopting an optimal search policy — Weitzman's Index Algorithm. We study both the seller's problem of best responding to his rivals, and the equilibria that arise in symmetric games among the senders.

While the model in its full generality is intricate and challenging for analytic approaches, we obtain concrete characterizations, with clear economic intuition, in special settings. Symmetric equilibria under concave priors are considerably more involved than under convex priors; see Appendix D for some partial results and technical insights. We believe that the setting is rich with potential for future studies. For example, part of the complexity in the model stems from the immense space of strategies of the advertisers. Finding meaningful restrictions on what an advertiser can reveal has the potential to make the model both more descriptive of practice and easier to harness. Another question open for future studies is the level of market efficiency achieved by competing advertisers. Our results in Section 5 show that, for small inspection cost and convex priors, the competition among the sellers essentially eliminates the inefficiency caused by search friction, but some of efficiency would remain when the inspection cost is higher.

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A Omitted Proofs in Section 2

A.1 Proof of Lemma 2

Proof of Lemma 2. ✓First, we show that the strategy space $\mathcal{G}(F,c)$ is convex given any prior F and cost c. Consider two feasible strategies $G \in \mathcal{G}(F,c)$ and $H \in \mathcal{G}(F,c)$ (with densities g and h). We want to prove that the convex combination $K(x) = \lambda \cdot G(x) + (1-\lambda) \cdot H(x)$ for any $x \in [0,1]$ (with density $\lambda \cdot g + (1-\lambda) \cdot h$) also belongs to space $\mathcal{G}(F,c)$ for any $\lambda \in [0,1]$. It is obvious that

$$\lambda \cdot \mathbb{E}_{v\sim G_{\cdot|\theta}}[\max\{v-\theta,0\}] + (1-\lambda) \cdot \mathbb{E}_{v\sim H_{\cdot|\theta}}[\max\{v-\theta,0\}] = c, \quad \forall \theta \in [-c, 1-c] \ ,$$

and

$$\lambda \cdot \int_{\theta=-c}^{1-c} g_{\cdot|v}(\theta) \ \mathrm{d}v + (1-\lambda) \cdot \int_{\theta=-c}^{1-c} h_{\cdot|v}(\theta) \ \mathrm{d}\theta = f(v), \quad \forall v \in [0,1] \ .$$

We have proved that any convex combination K also satisfies all the constraints of space $\mathcal{G}(F,c)$. Thus, the strategy space $\mathcal{G}(F,c)$ is convex.

Second, we show that the strategy space $\mathcal{G}(F,c)$ is also compact. We divide this proof into two steps. In the first step, we prove that the strategy space $\mathcal{G}(F,c)$ is bounded. The boundedness of space $\mathcal{G}(F,c)$ can be directly obtained by the fact that space $\mathcal{G}(F,c)$ is indeed a measure space over $[0,1]\times[-c,1-c]$. In the second step, we prove that the strategy space $\mathcal{G}(F,c)$ is closed, which means that the limit of any converging sequence inside this space also belongs to this space. We assume that sequence $\{G^m\}_{m\in\mathbb{Z}}$ weakly converges to a certain 2-D distribution G, and $G^m \in \mathcal{G}(F,c)$ for each $m \in \mathbb{Z}$ but $G \notin \mathcal{G}(F,c)$. This implies that G violates one or both constraints in Definition 4. First, we assume that there exists some $\tilde{\theta} \in [-c, 1-c]$ such that

$$\mathbb{E}_{v\sim G_{\cdot|\tilde{\theta}}}[\max\{v-\tilde{\theta},0\}] = d \neq c \ .$$

By the definition of weak convergence, the sequence of conditional distributions $\{G^m_{\cdot|\tilde{\theta}}\}_{m\in\mathbb{Z}}$ also weakly converges to the distribution $G_{\cdot|\theta}$. So for any $\epsilon > 0$, there exists $M > 0$ such that for any $m > M$ we have,

$$\mathbb{E}_{v\sim G^m_{\cdot|\tilde{\theta}}}[\max\{v-\tilde{\theta},0\}] \in (d-\epsilon, d+\epsilon) \ .$$

For sufficiently small $\epsilon > 0$ and for sufficiently large m, the distribution $G^m_{\cdot|\theta}$ also doesn't satisfy the first constraint, which forms a contradiction with the fact that $G^m \in \mathcal{G}(F,c)$. Second, we assume that there exists some $\tilde{v} \in [0,1]$ such that

$$\int_{-c}^{1-c} g_{\cdot|\theta}(\tilde{v}) \ \mathrm{d}v = e \neq f(\tilde{v}) \ .$$

Contradiction can be formed in the same manner. Thus we have proved that the limit point G must belong to space $\mathcal{G}(F,c)$, which makes the space closed. To sum up, we have achieved the boundedness and closeness of space $\mathcal{G}(F,c)$, and we can directly make space $\mathcal{G}(F,c)$ compact through the Heine-Borel Theorem. ■

B Supplementary Materials in Section 3

B.1 Proof of Theorem 1

✓Before proving Theorem 1, we have to introduce the following Lemmas 5 to 7 with respect to the properties of dual variables $\lambda(v)$ and $\mu(\theta)$.

Lemma 5. *For any dual solution (λ, μ), it holds that $\lambda(v) \geq u(v-c) \geq 0$ for each $v \in [0,1]$.*

Proof of Lemma 5. For each $v \in [0,1]$, by the fact that tuple $(v, v-c)$ satisfies Condition (6), we directly have $\lambda(v) \geq u(v-c) \geq 0$. ■

Lemma 6. *Assume there exists a constant $L > 0$ such that the interim utility function u is L-Lipschitz continuous over $[-c, 1-c]$. If there exists an optimal solution (λ^*, μ^*) to the dual problem $\mathcal{D}_{\mathsf{BR}}$, we can without loss assume that $\mu^*(\theta) \in [-L, 0]$ for each $\theta \in [-c, 1-c]$.*

Proof of Lemma 6. First, we prove $\mu^*(\theta) \leq 0$ for each $\theta \in [-c, 1-c]$ by contradiction. We assume that there exists an optimal dual solution (λ^*, μ^*) with $\mu^*(\overline{\theta}) > 0$ for some $\overline{\theta} \in [-c, 1-c]$. If $\lambda^*(v) > p(v, \overline{\theta}) - \mu^*(\overline{\theta})q(v, \overline{\theta})$ holds for all $v \in [\overline{\theta}, 1]$, then assuming $\mu^*(\overline{\theta}) = 0$ is without loss of generality, since this assumption does not affect the dual objective while keeping the constraints hold. If there exists a value $\overline{v} > \overline{\theta}$ with $\lambda^*(\overline{v}) = p(\overline{v}, \overline{\theta}) - \mu^*(\overline{\theta})q(\overline{v}, \overline{\theta})$, then for any $\theta \geq \overline{v}$, by Constraint (6), it holds that $\lambda^*(\overline{v}) \geq u(\overline{v})$. For the tuple $(\overline{v}, \overline{\theta})$, also by Constraint (6) and the monotonicity of u, we know that $\lambda^*(\overline{v}) = u(\overline{\theta}) - \mu^*(\overline{\theta})(\overline{v} - \theta) < u(\overline{\theta}) < u(\overline{v})$, which forms a contradiction. To sum up, we have proved that, for any optimal solution (λ^*, μ^*) to the dual problem, we can without loss assume that $\mu^*(\theta) \leq 0$ for each $\theta \in [-c, 1-c]$.

Second, we prove that for each $\theta \in [-c, 1-c]$, $\mu^*(\theta)$ is bounded by the constant L. The Lipschitz continuity of function u implies that $|u(x) - u(y)| \leq L \cdot |x-y|$ for any $x, y \in [-c, 1-c]$. We assume that there exists a $\theta_0 \in (-c, 1-c)$ (the case of $\theta_0 = -c$ or $1-c$ is trivial) with $\mu(\theta_0) < -L$. Then we know that $u(v-c) \geq -\mu(\theta_0)q(v, \theta_0) + p(v, \theta_0)$ for any $v \in [0, \theta_0 + c]$ and $u(v-c) \leq -\mu(\theta_0)q(v, \theta_0) + p(v, \theta_0)$ for any $v \in [\theta_0 + c, 1]$. So setting $\mu(\theta) = -L$ will weakly further decrease the objective function. To sum up, we have proved that $-\mu(\theta) \leq L$, that is function μ is bounded by the constant L. Directly by Constraint (6), we know that the function λ corresponding to the bounded function μ, is also bounded. ■

Lemma 7. *Assume there exists a constant $L > 0$ such that the interim utility function u is L-Lipschitz continuous over $[-c, 1-c]$. If there exists an optimal solution (λ^*, μ^*) to the dual problem $\mathcal{D}_{\mathsf{BR}}$, then the function $\lambda^*(\cdot)$ is non-decreasing and continuous over $[0,1]$.*

Proof of Lemma 7. Suppose (λ^*, μ^*) is optimal to the dual problem $\mathcal{D}_{\mathsf{BR}}$. By Constraint (6) and the objective of the dual problem, we have

$$\lambda^*(v) = \max_{\theta \in [-c, 1-c]} -\mu^*(\theta)q(v,\theta) + p(v,\theta), \quad \forall v \in [0,1] \ .$$

Given a certain $\theta \in [-c, 1-c]$, we define

$$\hat{L}_\theta(v) \triangleq -\mu^*(\theta)q(v,\theta) + p(v,\theta) = \begin{cases} -\mu^*(\theta)(v - \theta - c) + u(\theta) & \text{if } v \geq \theta \ , \\ u(v) + c\mu^*(\theta) & \text{if } v < \theta \ . \end{cases} \quad \forall v \in [0,1]$$

We observe that function $\hat{L}_\theta(\cdot)$ is linear over $[\theta, 1]$ and forms a shifted version of function u over $[0, \theta]$. Besides, $\lambda^*(v) = \max_{\theta\in[-c,1-c]} \hat{L}_\theta(v)$ for each $v \in [0, 1]$. By Lemma 6, it is obvious that function $\hat{L}_\theta$ is non-decreasing over $[0, 1]$ for each $\theta \in [-c, 1-c]$. By the fact that $\lambda^*(v) = \max_{\theta\in[-c,1-c]} \hat{L}_\theta(v)$ for each $v \in [0, 1]$, we have the function $\lambda^*(\cdot)$ is also non-decreasing over $[0, 1]$. Since the dual variable μ^* is a bounded function (Lemma 6) and the function u is continuous over $[-c, 1-c]$, we know that each $\hat{L}_\theta(\cdot)$ is continuous over $[0, 1]$ for any $\theta \in [-c, 1-c]$. This further implies that the function $\lambda^*(\cdot)$ is also continuous over $[0, 1]$. ∎

With these lemmas above, now we can prove Theorem 1.

Proof of Theorem 1. Based on the necessary conditions of the optimal solution to the dual problem (Lemmas 6 and 7), the dual problem $\mathcal{D}_{\mathsf{BR}}$ is equivalent to the following convex optimization problem $\mathcal{D}_{\mathsf{NEW}}$:

$$\min_{\lambda,\ \mu} \quad \int_0^1 \lambda(v) f(v) \ \mathrm{d}v \tag{$\mathcal{D}_{\mathsf{NEW}}$}$$

$$\text{subject to} \quad \lambda(v) = \max_{\theta\in[-c,1-c]} \{p(v, \theta) - \mu(\theta) q(v, \theta)\} \ , \quad \forall v \in [0, 1] \tag{8}$$

$$\mu(\theta) \le 0, \quad \forall \theta \in [-c, 1-c] \tag{9}$$

Any optimal solution (λ^*, μ^*) to the problem $\mathcal{D}_{\mathsf{NEW}}$ also forms an optimal solution to the dual problem $\mathcal{D}_{\mathsf{BR}}$, and vice versa. Thus, it suffices to prove that $\mathcal{D}_{\mathsf{NEW}}$ has an optimal solution, and for any optimal solution (λ^*, μ^*) it holds: (i) $\mu^*(\theta) \in [-L, 0]$ for any $\theta \in [-c, 1-c]$; and (ii) λ^* is non-decreasing and continuous over $[-c, 1-c]$. We divide the whole proof into two steps.

Step-1: There exists an optimal solution (λ^*, μ^*) to the problem $\mathcal{D}_{\mathsf{NEW}}$. Let $\mathrm{OPT} = \inf_{\lambda,\mu} \int_0^1 \lambda(v) f(v) \ \mathrm{d}v$ denote the optimal value of the dual problem. Let $\mathrm{OBJ}(\lambda, \mu)$ denote the objective value of any pair of feasible solutions (λ, μ). We aim to show that there exists a feasible solution (λ^*, μ^*) with $\mathrm{OBJ}(\lambda^*, \mu^*) = \mathrm{OPT}$. Whether the dual problem has an optimal solution or not, there always exists a feasible solution sequence $\{(\lambda^m, \mu^m)\}_{m\in\mathbb{Z}^+}$ such that the corresponding objective sequence $\{\mathrm{OBJ}(\lambda^m, \mu^m)\}_{m\in\mathbb{Z}^+}$ converges to the optimal value OPT^*.

By Lemma 7, we know that function λ is non-decreasing. Besides, without loss of generality, we can assume that function λ is also bounded. Combining these two facts, the sequence $\{\lambda^m\}_{m\in\mathbb{Z}^+}$ also weakly converges to certain function λ^* that satisfies $\int_0^1 \lambda^*(v) f(v) \ \mathrm{d}v = \mathrm{OPT}^*$. Next, we show that there exists a function μ^* that forms a feasible solution with the function λ^*. For any $m \in \mathbb{Z}^+$, each pair (λ^m, μ^m) is feasible. So by Constraint (6), for any $\theta \in [-c, 1-c]$, we have that

$$\max_{v\in[\theta+c,1]} \frac{u(\theta) - \lambda^m(v)}{v - \theta - c} \le \mu^m(\theta) \le \min\left\{ \min_{v\in[0,\theta)} \frac{\lambda^m(v) - u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta) - \lambda^m(v)}{v - \theta - c} \right\} \ .$$

Since the pair (λ^m, μ^m) is feasible, we have

$$\max_{v\in[\theta+c,1]} \frac{u(\theta) - \lambda^m(v)}{v - \theta - c} \le \min\left\{ \min_{v\in[0,\theta)} \frac{\lambda^m(v) - u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta) - \lambda^m(v)}{v - \theta - c} \right\} \ ,$$

and such μ^m exists. When $m \to \infty$, we have that

$$\lim_{m\to\infty} \max_{v\in[\theta+c,1]} \frac{u(\theta) - \lambda^m(v)}{v - \theta - c} = \max_{v\in[\theta+c,1]} \frac{u(\theta) - \lambda^*(v)}{v - \theta - c} \ ,$$

$$
\begin{aligned}
&\lim_{m\to\infty} \min\left\{ \min_{v\in[0,\theta)} \frac{\lambda^m(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda^m(v)}{v-\theta-c} \right\} \\
&= \min\left\{ \min_{v\in[0,\theta)} \frac{\lambda^*(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda^*(v)}{v-\theta-c} \right\} .
\end{aligned}
$$

Thus, we know that there also exists such μ^* that forms a feasible solution along with function λ^*. To sum up, we have proved that there exists an optimal solution (λ^*, μ^*) to the problem $\mathcal{D}_{\mathsf{NEW}}$. By the equivalence of $\mathcal{D}_{\mathsf{NEW}}$ and $\mathcal{D}_{\mathsf{BR}}$, there also exists an optimal solution to the dual problem $\mathcal{D}_{\mathsf{BR}}$.

Step-2: For any optimal solution (λ^*, μ^*) to the dual problem $\mathcal{D}_{\mathsf{BR}}$, $\mu^*(\theta) \in [-L, 0]$ for any $\theta \in [-c, 1-c]$, and λ^* is non-decreasing and continuous over $[-c, 1-c]$. These conditions hold directly by Lemmas 6 and 7. ■

B.2 Proof of Theorem 2

Proof of Theorem 2. First, we prove the sufficiency of Theorem 2. The dual variables $\lambda(v)$ and $\mu(\theta)$ are both bounded measurable functions, so the order of integration can be interchanged for any double integral over $[0,1] \times [-c, 1-c]$ under Fubini's Theorem. It is obvious that both the primal and the dual have a feasible solution. We consider any pair of feasible solutions (g, λ, μ). First by Constraint (4), we have that

$$
\int_{v=0}^{1} \int_{\theta=-c}^{1-c} \lambda(v) g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta = \int_{0}^{1} \lambda(v) f(v) \ \mathrm{d}v \ . \tag{10}
$$

By Constraint (5), we have that

$$
\int_{v=0}^{1} \int_{\theta=-c}^{1-c} \mu(\theta) q(v,\theta) g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta = \int_{\theta=-c}^{1-c} \int_{v=0}^{1} \mu(\theta) q(v,\theta) g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta = 0 \ . \tag{11}
$$

After combining Equation (10) and Equation (11), we achieve that

$$
\int_{v=0}^{1} \int_{\theta=-c}^{1-c} \left[\lambda(v) + \mu(\theta) q(v,\theta)\right] \ g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta = \int_{0}^{1} \lambda(v) f(v) \ \mathrm{d}v \ . \tag{12}
$$

By Constraint (6), we have that

$$
\int_{v=0}^{1} \int_{\theta=-c}^{1-c} [\lambda(v) + \mu(\theta) q(v,\theta)] g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta \geq \int_{v=0}^{1} \int_{\theta=-c}^{1-c} p(v,\theta) g(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta \ . \tag{13}
$$

Next, we assume that there exists a pair of feasible solutions (g^*, λ^*, μ^*) that satisfies Condition (7), that is

$$
\int_{v=0}^{1} \int_{\theta=-c}^{1-c} [\lambda^*(v) + \mu^*(\theta) q(v,\theta)] g^*(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta = \int_{v=0}^{1} \int_{\theta=-c}^{1-c} p(v,\theta) g^*(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta \ . \tag{14}
$$

For this special pair of feasible solutions, after combining Equation (12) and Equation (14), then we have

$$
\int_{0}^{1} \lambda^*(v) f(v) \ \mathrm{d}v = \int_{v=0}^{1} \int_{\theta=-c}^{1-c} p(v,\theta) g^*(v,\theta) \ \mathrm{d}v \ \mathrm{d}\theta \ . \tag{15}
$$

For any feasible solution g to the primal problem, after combining Equation (12) and Equation (13), we obtain that

$$\int_0^1 \lambda^*(v) f(v)\ \mathrm{d}v \geq \int_{v=0}^1 \int_{\theta=-c}^{1-c} p(v,\theta) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ . \tag{16}$$

By Equation (15) and Inequality (16), we have

$$\int_{v=0}^1 \int_{\theta=-c}^{1-c} p(v,\theta) g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta \geq \int_{v=0}^1 \int_{\theta=-c}^{1-c} p(v,\theta) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ ,$$

which means the special solution g^* is optimal to the primal problem $\mathcal{P}_{\mathsf{BR}}$. In the same manner, we can also prove that (λ^*, μ^*) is the optimal solution to the dual problem $\mathcal{D}_{\mathsf{BR}}$. Besides, Equation (15) implies that the strong duality holds between this primal problem and the dual problem.

Then we prove the necessity of Theorem 2. We assume that G^* with density g^* is an optimal solution to $\mathcal{P}_{\mathsf{BR}}$ and (λ^*, μ^*) is the optimal solution to $\mathcal{D}_{\mathsf{BR}}$. Since the strong duality holds, we know that

$$\int_0^1 \lambda^*(v) f(v)\ \mathrm{d}v = \int_{v=0}^1 \int_{\theta=-c}^{1-c} p(v,\theta)\ g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ . \tag{17}$$

By Constrain (4), we have that

$$\int_{v=0}^1 \int_{\theta=-c}^{1-c} \lambda^*(v) g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_0^1 \lambda^*(v) f(v)\ \mathrm{d}v\ . \tag{18}$$

By Constrain (5), we have

$$\int_{v=0}^1 \int_{\theta=-c}^{1-c} \mu^*(\theta) q(v,\theta)\ g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = 0\ . \tag{19}$$

Combining Equation (18) and Equation (19), we have achieve that

$$\int_{v=0}^1 \int_{\theta=-c}^{1-c} [\lambda^*(v) + \mu^*(\theta) q(v,\theta)]\ g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_0^1 \lambda^*(v) f(v)\ \mathrm{d}v\ . \tag{20}$$

Combining Equation (17) and Equation (12), we achieve that

$$\int_{v=0}^1 \int_{\theta=-c}^{1-c} [\lambda^*(v) + \mu^*(\theta) q(v,\theta) - p(v,\theta)]\ g^*(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = 0\ ,$$

which means that the optimal solution to the primal problem G^* and the optimal solution to the dual problem (λ^*, μ^*) satisfy Condition (7). ■

B.3 Algorithm 2 and Proof of Theorem 3

Proof of Lemma 3. For any feasible 2-D distribution G, there being no such threshold value $\underline{v}$ implies that, there exists a tuple $(0, \theta) \in \mathrm{supp}(G)$ such that $\theta > 0$. In any best response G^*, without loss of generality, we can assume that such $(0, \theta) \in \mathrm{supp}(G)$ with $\theta > 0$ does not exist since pooling value 0 up to a positive index brings no utility increase. ■

Here is the algorithm to construct the dual variable $\mu(\theta)$ for any $\theta \in [-c, 1-c]$.

Algorithm 2 Constructions of the dual variable μ

Input: Strategy G; Interim utility u; Cost c; Threshold value $\underline{v}$; All $\lambda(v)$ for $v \in [0,1]$.
Output: If the algorithm

for $\theta \in [-c, \underline{v} - c]$ **do**
  $\mu(\theta) \leftarrow 0$
for $\theta \in \mathrm{supp}(G_\theta) \cup (\underline{v} - c, 1-c]$ **do**
  if $\mathrm{supp}(G_{\cdot|\theta}) = \{\theta + c\}$ **then**
    if $\max_{v \in (\theta+c,1]} \frac{u(\theta)-\lambda(v)}{v-\theta-c} \le \min\left\{\min_{v\in[0,\theta)} \frac{\lambda(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda(v)}{v-\theta-c}\right\}$ **then**
      return False
    $\mu(\theta) \leftarrow \min\left\{\min_{v\in[0,\theta)} \frac{\lambda(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda(v)}{v-\theta-c}\right\}$
  else
    $\mu_{\mathrm{ref}} \leftarrow$ None
    for $v \in \mathrm{supp}(G_{\cdot|\theta}) \setminus \{\theta + c\}$ **do**
      $\mu_{\mathrm{cur}} \leftarrow \dfrac{p(v,\theta) - \lambda(v)}{q(v,\theta)}$
      if $\mu_{\mathrm{ref}} =$ None **then**
        $\mu_{\mathrm{ref}} \leftarrow \mu_{\mathrm{cur}}$
      else if $\mu_{\mathrm{cur}} \neq \mu_{\mathrm{ref}}$ **then**
        return False
    $\mu(\theta) \leftarrow \mu_{\mathrm{ref}}$
for $\theta \notin \mathrm{supp}(G_\theta)$ **do**
  if $\max_{v \in (\theta+c,1]} \frac{u(\theta)-\lambda(v)}{v-\theta-c} \le \min\left\{\min_{v\in[0,\theta)} \frac{\lambda(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda(v)}{v-\theta-c}\right\}$ **then**
    return False
  $\mu(\theta) \leftarrow \min\left\{\min_{v\in[0,\theta)} \frac{\lambda(v)-u(v)}{c}, \min_{v\in[\theta,\theta+c)} \frac{u(\theta)-\lambda(v)}{v-\theta-c}\right\}$
return True

Before proving Theorem 3, we have to introduce the following lemmas.

Lemma 8 (Case 1 in Algorithm 1)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, then we without loss of generality assume that $\lambda^*(v) = 0$ for each $v \in [0, \underline{v}]$ and $\mu^*(\theta) = 0$ for each $\theta \in [-c, \underline{v} - c]$.*

Proof of Lemma 8. Since strategy G^* is optimal to the primal problem, the threshold value $\underline{v}$ is above zero. Assuming no pooling is made to values $[0, \underline{v})$ is without loss of generality, thus we have $\lambda^*(v) = 0$ for any $v \in [0, \underline{v}]$. Besides, we can assume that $\mu^*(\theta) = 0$ for each $\theta \in [-c, \underline{v} - c]$ without loss of generality. Since assuming $\mu^*(\theta) = 0$ for each $\theta \in [-c, \underline{v} - c]$ does not affect the objective of the dual while keeping all constraints held. ■

Lemma 9 (Case 2 in Algorithm 1)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, if there exists $v \in [\underline{v}, 1]$ such that $(v, v-c) \in \mathrm{supp}(G)$, then $\lambda^*(v) = u(v-c)$.*

Proof of Lemma 9. By Theorem 2, it directly holds that $\lambda^*(v) = -\mu^*(v-c)q(v, v-c) + p(v, v-c) = p(v, v-c) = u(v-c)$. ■

Lemma 10 (Case 3.1 in Algorithm 1)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any (v, θ) with $v = \theta + c$, if there is a sequence $\{(v^m, \theta^m)\}_{m \in \mathbb{Z}^+}$ converging to*

(v,θ) such that : (i) $(v^m,\theta^m) \in \operatorname{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone; and (iv) $v^m \in (\theta^m, v)$ for any $m \in \mathbb{Z}^+$, then $\lambda^*(v) = u(v-c)$.

Proof of Lemma 10. For any $m \in \mathbb{Z}^+$, since the tuple $(v^m,\theta^m) \in \operatorname{supp}(G^*)$, Theorem 2 implies that

$$\lambda^*(v^m) = -\mu^*(\theta^m)q(v^m,\theta^m) + p(v^m,\theta^m) = -\mu^*(\theta^m)(v^m - \theta^m - c) + u(\theta^m)\ .$$

Besides, since the function u is Lipschitz continuous everywhere, we have already proved the boundedness of the function u in Theorem 1. Thus, we know that

$$\lambda^*(v) = \lim_{m\to\infty} \lambda^*(v^m) = \lim_{m\to\infty} p(v^m,\theta^m) = u(v-c)\ .$$

■

Lemma 11 (Case 3.2 in Algorithm 1 for the limit point (v,θ) with $v < \theta$)**.** *Given G^* and (λ^*,μ^*) are both optimal to the primal and dual problems, for any (v,θ) with $v \in (\underline{v},\theta)$, if there is a sequence $\{(v^m,\theta^m)\}_{m\in\mathbb{Z}^+}$ converging to (v,θ) such that : (i) $(v^m,\theta^m) \in \operatorname{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone; and (iv) $\theta^m > v$ for any $m \in \mathbb{Z}^+$, then $-\mu^*(\theta^{m_1}) = u'(\theta^{m_1}) = u'(\theta^{m_2}) = -\mu^*(\theta^{m_2})$ for any $m_1, m_2 \in \mathbb{Z}^+$. Besides, $\lambda^*(v) = u'(\theta^1)q(v,\theta) + p(v,\theta)$.*

Proof of Lemma 11. Without loss of generality, we can assume that $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is also strictly increasing. For any $m \in \mathbb{Z}^+$, by Theorem 2 and the fact that function λ^* is continuous, we know that for any $m_1, m_2 \in \mathbb{Z}^+$ with $m_1 > m_2$,

$$\lambda^*(v^{m_1}) = (-c) \max_{\theta'\in[v,1-c]} -\mu^*(\theta') + u(v^{m_1}) = -\mu^*(\theta^{m_1})(-c) + u(v^{m_1})\ ,$$

and

$$\lambda^*(v^{m_2}) = (-c) \max_{\theta'\in[v,1-c]} -\mu^*(\theta') + u(v^{m_2}) = -\mu^*(\theta^{m_2})(-c) + u(v^{m_2})\ .$$

Combining these two equations, we achieve that

$$\max_{\theta'\in[v,1-c]} -\mu^*(\theta') = -\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2})\ .$$

Next, we aim to prove that $\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2}) = u'(\theta^{m_1}) = u'(\theta^{m_2})$. To form feasible indices θ^{m_1} and θ^{m_2}, there must exist $(\hat{v}^{m_1},\theta^{m_1}) \in \operatorname{supp}(G^*)$ with $\hat{v}^{m_1} > \theta^{m_1} + c$, and $(\hat{v}^{m_2},\theta^{m_2}) \in \operatorname{supp}(G^*)$ with $\hat{v}^{m_2} > \theta^{m_2} + c$. By Theorem 2, it holds that

$$\lambda^*(\hat{v}^{m_1}) = -\mu^*(\theta^{m_1})(\hat{v}^{m_1} - \theta^{m_1} - c) + u(\theta^{m_1}) \geq -\mu^*(\theta^{m_2})(\hat{v}^{m_1} - \theta^{m_2} - c) + u(\theta^{m_2})\ ,$$

and

$$\lambda^*(\hat{v}^{m_2}) = -\mu^*(\theta^{m_2})(\hat{v}^{m_2} - \theta^{m_2} - c) + u(\theta^{m_2}) \geq -\mu^*(\theta^{m_1})(\hat{v}^{m_2} - \theta^{m_1} - c) + u(\theta^{m_1})\ .$$

Since (i) the function $-\mu^*(\theta^{m_1})q(\ \cdot\ ,\theta^{m_1}) + u(\theta^{m_1})$ is linear over $[\theta_1, 1-c]$ and crosses $(\theta^{m_1} + c, u(\theta^{m_1}))$; (ii) the function $-\mu^*(\theta^{m_2})q(\ \cdot\ ,\theta^{m_2}) + u(\theta^{m_2})$ is linear over $[\theta_2, 1-c]$ and crosses $(\theta^{m_2} + c, u(\theta^{m_2}))$; and (iii) $-\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2})$, it must hold that $\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2}) = u'(\theta^{m_1}) = u'(\theta^{m_2})$ and these two functions overlap over $[\theta_2 + c, 1-c]$.

Recall that λ^*, p, q are all continuous; we complete our proof by showing that

$$\lambda^*(v) = \lim_{m\to\infty} \lambda^*(v^m) = \lim_{m\to\infty} -\mu^*(\theta^m)q(v^m,\theta^m) + p(v^m,\theta^m) = u'(\theta^1)q(v,\theta) + p(v,\theta) \ .$$

■

Lemma 12 (Case 3.2 in Algorithm 1 for the limit point (v,θ) with $v \in (\theta, \theta + c)$)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any (v,θ) with $v \in (\max\{\underline{v}, \theta\}, \theta + c)$, if there is a sequence $\{(v^m, \theta^m)\}_{m\in\mathbb{Z}^+}$ converging to (v,θ) such that : (i) $(v^m,\theta^m) \in \mathrm{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone; (iv) let $\theta' = \frac{v+\theta}{2}$, $\theta^m \in (v-c, \theta')$ and $v^m \in (\theta', v)$ for any $m \in \mathbb{Z}^+$, then $-\mu^*(\theta^{m_1}) = u'(\theta^{m_1}) = u'(\theta^{m_2}) = -\mu^*(\theta^{m_2})$ for any $m_1, m_2 \in \mathbb{Z}^+$. Besides, $\lambda^*(v) = u'(\theta^1)q(v,\theta) + p(v,\theta)$.*

Proof of Lemma 12. Without loss of generality, we can assume that $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is also strictly increasing. We can consider two tuples (v^3,θ^3), (v^4,θ^4) with $\theta' \le v_3 < v_4 < v$ and $\theta_4 > \theta_3$. There exist tuples $(\bar{v}^3,\theta^3), (\bar{v}^4,\theta^4) \in \mathrm{supp}(G)$ such that $\bar{v}^3 > \theta^3 + c$ and $\bar{v}^4 > \theta^4 + c$. Based on Theorem 2, we achieve the following inequalities:

$$\begin{aligned}
-\mu^*(\theta^3)q(v^3,\theta^3) + p(v^3,\theta^3) &\ge -\mu^*(\theta^4)q(v^3,\theta^4) + p(v^3,\theta^4) \ , \\
-\mu^*(\theta^3)q(\bar{v}^3,\theta^3) + p(\bar{v}^3,\theta^3) &\ge -\mu^*(\theta^4)q(\bar{v}^3,\theta^4) + p(\bar{v}^3,\theta^4) \ , \\
-\mu^*(\theta^4)q(v^4,\theta^4) + p(v^4,\theta^4) &\ge -\mu^*(\theta^3)q(v^4,\theta^3) + p(v^4,\theta^3) \ , \\
-\mu^*(\theta^4)q(\bar{v}^4,\theta^4) + p(\bar{v}^4,\theta^4) &\ge -\mu^*(\theta^3)q(\bar{v}^4,\theta^3) + p(\bar{v}^4,\theta^3) \ .
\end{aligned}$$

Since (i) the function $-\mu^*(\theta^3)q(\,\cdot\,,\theta^3)+p(\,\cdot\,,\theta^3)$ is linear over (θ', v); (ii) the function $-\mu^*(\theta^3)q(\,\cdot\,,\theta^3)+p(\,\cdot\,,\theta^3)$ crosses $(\theta^3+c, u(\theta^3))$; (iii) the function $-\mu^*(\theta^4)q(\,\cdot\,,\theta^4)+p(\,\cdot\,,\theta^4)$ is linear over (θ', v); and (iv) the function $-\mu^*(\theta^4)q(\,\cdot\,,\theta^4)+p(\,\cdot\,,\theta^4)$ crosses $(\theta^4+c, u(\theta^4))$, the above four inequalities imply that $\mu^*(\theta^3) = \mu^*(\theta^4) = u'(\theta^3) = u'(\theta^4)$ and these two functions overlap over $(\theta', 1]$. The above analysis holds for any $m_1, m_2 \in \mathbb{Z}^+$ with $m_1 < m_2$. Thus we know that $\mu^*(\theta^{m_1}) = \mu^*(\theta^{m_2}) = u'(\theta^{m_1}) = u'(\theta^{m_2})$ for any $m_1, m_2 \in \mathbb{Z}^+$. For any $m \in \mathbb{Z}^+$, it holds

$$\lambda^*(v^m) = p(v^m,\theta^m) - \mu^*(\theta^m)q(v^m,\theta^m) \ .$$

Passing m to infinity, we achieve that

$$\lambda^*(v) = p(v,\theta) - \mu^*(\theta^1)q(v,\theta) = u'(\theta^1)q(v,\theta) + p(v,\theta) \ .$$

■

Lemma 13 (Case 3.2 in Algorithm 1 for the limit point (v,θ) with $v > \theta + c$)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any (v,θ) with $v > \theta + c$, if there is a sequence $\{(v^m,\theta^m)\}_{m\in\mathbb{Z}^+}$ converging to (v,θ) such that : (i) $(v^m,\theta^m) \in \mathrm{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone; (iv) let $\theta' = \frac{v-c+\theta}{2} \in (\theta, v-c)$, $\theta^m \in [0, \theta')$ and $v^m \in (\theta'+c, v)$ for any $m \in \mathbb{Z}^+$, then $-\mu^*(\theta^{m_1}) = u'(\theta^{m_1}) = u'(\theta^{m_2}) = -\mu^*(\theta^{m_2})$ for any $m_1, m_2 \in \mathbb{Z}^+$. Besides, $\lambda^*(v) = u'(\theta^1)q(v,\theta) + p(v,\theta)$.*

Proof of Lemma 13. We proceed with the proof in two steps.

Step-1: Prove that $-\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2}) = u'(\theta^{m_1}) = u'(\theta^{m_2})$ for any $m_1, m_2 \in \mathbb{Z}^+$. Let $v_l^m \triangleq \inf \mathrm{supp}(G^*_{\cdot|\theta^m})$ for any $m \in \mathbb{Z}^+$. Since the sequence $\{(v^m,\theta^m)\}_{m\in\mathbb{Z}^+}$ converges to (v,θ), the

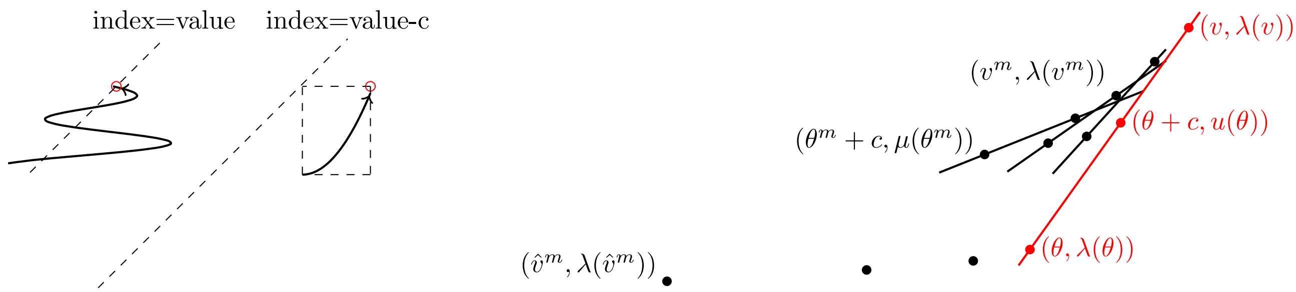

Figure 4: Graph illustration of Lemma 13

sequence $\{(v_l^m, \theta^m)\}_{m\in\mathbb{Z}^+}$ also converges to some (v_l, θ) with $v_l < \theta + c$. We divide the proof into three cases.

Case 1: $v_l < \theta$**.** We can prove this result using a similar method as Lemma 11.

Case 2: $v_l \in (\theta, \theta + c)$**.** We can prove this result using a similar method as Lemma 12.

Case 3: $v_l = \theta$**.** If the sequence $\{\theta^m\}$ is also decreasing, then we know that there exists a subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ such that $\{(v_l^{m(k)}, \theta^{m(k)})\}_{k\in\mathbb{Z}^+}$ converges to (v_l, θ), and is decreasing. In this case, we can prove the target result using a similar method as Lemma 11.

Thus, without loss, we can assume that the sequence $\{\theta^m\}$ is increasing, and there exists a subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ such that $\{(v_l^{m(k)}, \theta^{m(k)})\}_{k\in\mathbb{Z}^+}$ converges to (v_l, θ), and is also increasing. Abusing the notations a little, we rename the sequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ as $\{m\}_{m\in\mathbb{Z}^+}$. Since the sequences $\{v^m\}_{m\in\mathbb{Z}^+}$ and $\{\theta^m\}_{m\in\mathbb{Z}^+}$ are both increasing, by Theorem 2, we know that $\{-\mu^*(\theta^m)\}_{m\in\mathbb{Z}^+}$ is also increasing. We have already prove the boundedness of μ^*, thus $\{-\mu^*(\theta^m)\}_{m\in\mathbb{Z}^+}$ also converges to some real number, which is denoted as $-\hat{\mu}$, that is $\lim_{m\to\infty} -\mu^*(\theta^m) = -\hat{\mu}$.

Next, we aim to prove that there exists an $M > 0$ such that for any $m > M$, $-\mu^*(\theta^m) = -\hat{\mu}$. We prove it through contradiction. We assume that for any $m < \infty$, it holds that $-\mu^*(\theta^m) < -\hat{\mu}$. We consider three sequences: (i) $\{(v_l^m, \lambda(v_l^m))\}_{m\in\mathbb{Z}^+}$, (ii) $\{(\theta^m + c, u(\theta^m))\}_{m\in\mathbb{Z}^+}$, and (iii) $\{(v^m, \lambda(v^m))\}_{m\in\mathbb{Z}^+}$. Let (x_a, y_a) denote the limit point of the sequence $\{(v_l^m, \lambda(v_l^m))\}_{m\in\mathbb{Z}^+}$. Let (x_b, y_b) denote the limit point of the sequence $\{(\theta^m + c, u(\theta^m))\}_{m\in\mathbb{Z}^+}$. Let (x_c, y_c) denote the limit point of the sequence $\{(v^m, \lambda(v^m))\}_{m\in\mathbb{Z}^+}$. Based on the graph illustrations, we can observe that $u'(\theta) \le -\hat{\mu}$. Next we aim to show that these three points $(x_a, y_a), (x_b, y_b), (x_c, y_c)$ lie in a common linear function. First, we show that $\frac{y_c - y_b}{x_c - x_b} = -\hat{\mu}$:

$$
\begin{aligned}
\frac{y_c - y_b}{x_c - x_b} &= \lim_{m\to\infty} \frac{\lambda^*(v^m) - u(\theta^m)}{v^m - (\theta^m + c)} \\
&= \lim_{m\to\infty} \frac{-\mu^*(\theta^m)[v^m - (\theta^m + c)]}{v^m - (\theta^m + c)} \\
&= \lim_{m\to\infty} -\mu^*(\theta^m) = -\hat{\mu} \ .
\end{aligned}
$$

Next, we prove that $\frac{y_b - y_a}{x_b - x_a} = -\hat{\mu}$. For any $m \in \mathbb{Z}^+$, we have that

$$\frac{y_b - y_a}{x_b - x_a} = \lim_{m\to\infty} \frac{u(\theta^m) - \lambda(v_l^m)}{\theta^m + c - v_l^m}$$
$$= \lim_{m\to\infty} \frac{u(\theta^m) - (-\mu^*(\theta^m)q(v_l^m, \theta^m) + p(v_l^m, \theta^m))}{\theta^m + c - (v_l^m - \theta^m) - \theta^m} = -\hat{\mu} \ .$$

Combining these two facts, we know that $(x_a, y_a), (x_b, y_b), (x_c, y_c)$ lie in a common linear function. Under this fact, we know that for sufficiently small $\epsilon > 0$,

$$-\mu^*(\theta - \epsilon)q(v_l, \theta - \epsilon) + p(v_l, \theta - \epsilon) > -\mu^*(\theta)q(v_l, \theta) + p(v_l, \theta) \ ,$$

which forms a contradiction with the fact that the sequence $\{v_l^m, \theta^m\}_{m\in\mathbb{Z}^+}$ converges to (v_l, θ) with $v_l = \theta$. Thus, we have proved that the assumption in the beginning is invalid. To sum up, we have proved that there exists a sufficiently large $M > 0$ such that $-\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2})$ for any $m_1, m_2 > M$. Using a similar idea of Lemma 11, we can also prove that $-\mu^*(\theta^{m_1}) = -\mu^*(\theta^{m_2}) = u'(\theta^{m_1}) = u'(\theta^{m_2})$ for any $m_1, m_2 > M$.

Step-2: Prove that $\lambda^*(v) = u'(\theta^1)q(v, \theta) + p(v, \theta)$. For any $m \in \mathbb{Z}^+$, Theorem 2 implies that

$$\lambda^*(v^m) = p(v^m, \theta^m) - \mu^*(\theta^m)q(v^m, \theta^m) \ .$$

Passing m to infinity, since functions λ^*, p, q are both continuous, we achieve that

$$\lambda^*(v) = p(v, \theta) - \mu^*(\theta^1)q(v, \theta) = u'(\theta^1)q(v, \theta) + p(v, \theta) \ .$$

■

Lemma 14 (Case 3.2 in Algorithm 1 for the limit point (v, θ) with $v = \theta$)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any (v, θ) with $v = \theta + c$, if there is a sequence $\{(v^m, \theta^m)\}_{m\in\mathbb{Z}^+}$ converging to (v, θ) such that : (i) $(v^m, \theta^m) \in \mathrm{supp}(G)$ for any $m \in \mathbb{Z}^+$; (ii) $\{v^m\}_{m\in\mathbb{Z}^+}$ is strictly increasing; and (iii) $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is monotone, then there exists a sufficiently large $M > 0$ such that $-\mu^*(\theta^m) = u'(\theta)$ for any $m > M$. Besides, $\lambda^*(v) = u'(\theta)q(v, \theta) + p(v, \theta)$.*

Proof of Lemma 14. Based on the properties of the sequence $\{(v^m, \theta^m)\}_{m\in\mathbb{Z}^+}$, we can divide the proof into three cases:

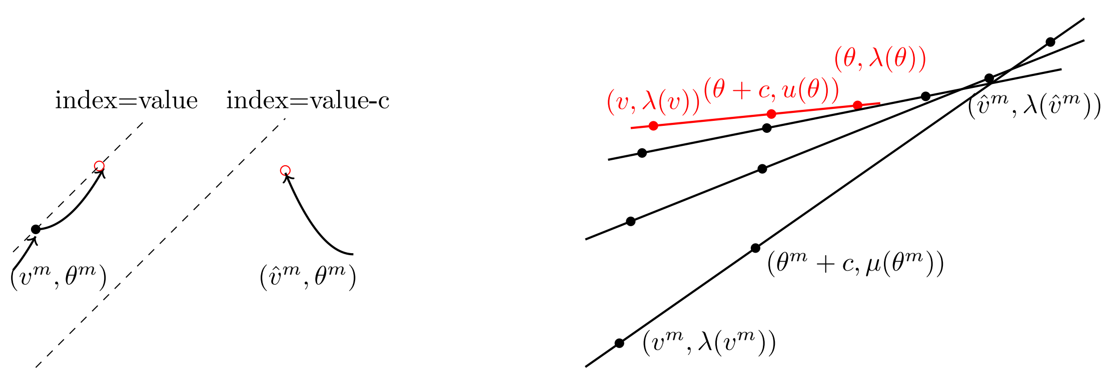

Figure 5: Graph illustration of Lemma 14

Case 1: The sequence $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is decreasing. This lemma can be proved in a method similar to that of Lemma 11.

Case 2: The sequence $\{\theta^m\}_{m\in\mathbb{Z}^+}$ is increasing, and there exists $M > 0$ such that $v^m \geq \theta^m$ for any $m > M$. For any $m \in \mathbb{Z}^+$, there exists $(\hat{v}^m, \theta^m) \in \text{supp}(G^*)$ with $\hat{v}^m > \theta^m + c$. By Theorem 2, we have that

$$p(\hat{v}^m, \theta^m) - \mu^*(\theta^m)q(\hat{v}^m, \theta^m) \geq p(\hat{v}^m, \theta^{m+1}) - \mu^*(\theta^{m+1})q(\hat{v}^m, \theta^{m+1}) \ .$$

Since the functions $p(\cdot, \theta^m) - \mu^*(\theta^m)q(\cdot, \theta^m)$ and $p(\cdot, \theta^{m+1}) - \mu^*(\theta^{m+1})q(\cdot, \theta^{m+1})$ are both linear over $[\hat{v}^m, 1]$, it follows that $-\mu^*(\theta^m) \geq -\mu^*(\theta^{m+1})$ and $\hat{v}^m \geq \hat{v}^{m+1}$. Since the function μ^* is bounded (Theorem 1), we know that $\lim_{m\to\infty} -\mu^*(\theta^m) = -\underline{\mu}$.

We consider three sequences: (i) $\{(v^m, \lambda(v^m))\}_{m\in\mathbb{Z}^+}$, (ii) $\{(\theta^m + c, u(\theta^m))\}_{m\in\mathbb{Z}^+}$, and (iii) $\{(\hat{v}^m, \lambda(\hat{v}^m))\}_{m\in\mathbb{Z}^+}$. Notice that for any $m \in \mathbb{Z}^+$, three points $(v^m, \lambda(v^m)), (\theta^m + c, u(\theta^m)), (\hat{v}^m, \lambda(\hat{v}^m))$ lie in a common linear function $-\mu^*(\theta^m)q(\cdot, \theta^m) + p(\cdot, \theta^m)$ over $[\theta^m, 1 - c]$. Let (x_a, y_a) denote the limit point of the sequence $\{(v^m, \lambda(v^m))\}_{m\in\mathbb{Z}^+}$. Let (x_b, y_b) denote the limit point of the sequence $\{(\theta^m + c, u(\theta^m))\}_{m\in\mathbb{Z}^+}$. Let (x_c, y_c) denote the limit point of the sequence $\{(\hat{v}^m, \lambda(\hat{v}^m))\}_{m\in\mathbb{Z}^+}$. We can observe that the function $-\mu^*(\theta)q(\cdot, \theta) + p(\cdot, \theta)$ crosses the point (x_b, y_b). It follows that $u'(\theta) \geq -\underline{\mu}$.

Next we aim to prove $u'(\theta) = -\underline{\mu}$ through contradiction. We assume that $u'(\theta) > -\underline{\mu}$. We divide the proof into two cases.

Case 1: Points (x_b, y_b) and (x_c, y_c) do not overlap. Next, we show that $\frac{y_c - y_b}{x_c - x_b} = -\underline{\mu}$.

$$\begin{aligned}\frac{y_c - y_b}{x_c - x_b} &= \lim_{m\to\infty} \frac{\lambda^*(\hat{v}^m) - u(\theta^m)}{\hat{v}^m - (\theta^m + c)} \\ &= \lim_{m\to\infty} \frac{-\mu^*(\theta^m)[\hat{v}^m - (\theta^m + c)]}{\hat{v}^m - (\theta^m + c)} \\ &= \lim_{m\to\infty} -\mu^*(\theta^m) = -\underline{\mu} \ .\end{aligned}$$

In a similar manner, we can also prove that $\frac{y_b - y_a}{x_b - x_a} = -\underline{\mu}$. Thus, we have proved that points $(x_a, y_a), (x_b, y_b), (x_c, y_c)$ lie in a common linear function. Next we consider an index $\theta + \epsilon$ for sufficiently small $\epsilon > 0$. Since the function λ^* is continuous, we know that

$$-\mu^*(\theta)q(x_c, \theta) + p(x_c, \theta) \geq -\mu^*(\theta + \epsilon)q(x_c, \theta + \epsilon) + p(x_c, \theta + \epsilon) \ ,$$

which further implies that $-\mu^*(\theta + \epsilon) < -\underline{\mu}$. Finally, we know that

$$-\mu^*(\theta + \epsilon)q(v, \theta + \epsilon) + p(v, \theta + \epsilon) = u(v) + \mu^*(\theta + \epsilon) \cdot c > u(v) + \underline{\mu} \cdot c = \lambda(v) \ ,$$

which forms a contradiction to Theorem 2. Thus, we have proved that $u'(\theta) = -\underline{\mu}$.

Case 2: Points (x_b, y_b) and (x_c, y_c) overlap. For any $m \in \mathbb{Z}^+$, it holds $\lambda(\hat{v}^m) \geq u(\hat{v}^m - c)$. The fact that points (x_b, y_b) and (x_c, y_c) overlap implies that $\lambda^*(\theta + c) = u(\theta)$. Further, we have that

$$\frac{\lambda(\hat{v}^m) - \lambda(\theta + c)}{\hat{v}^m - \theta - c} \geq \frac{u(\hat{v}^m - c) - u(\theta)}{\hat{v}^m - \theta - c} \ .$$

Based on the graph illustrations of these functions, we know $-\mu^*(\theta^m) \geq \frac{\lambda(\hat{v}^m) - \lambda(\theta + c)}{\hat{v}^m - \theta - c}$. Thus, we have that

$$-\mu^*(\theta^m) \geq \frac{u(\hat{v}^m - c) - u(\theta)}{\hat{v}^m - \theta - c} \ .$$

Passing m to infinity, we achieve that

$$-\underline{\mu} = \lim_{m \to \infty} -\mu^*(\theta^m) \geq u'(\theta) \ ,$$

which forms a contradiction with the former assumption.

By combining these two cases, we have proved that $-\underline{\mu} = u'(\theta)$. We know that $\frac{y_b - y_a}{x_b - x_a} = -\underline{\mu}$. Thus, we have that

$$u'(\theta) = -\underline{\mu} = \frac{y_b - y_a}{x_b - x_a} = \frac{u(\theta) - \lambda^*(v)}{\theta + c - v} \ ,$$

which implies that $\lambda^*(v) = u(\theta) + u'(\theta)(v - \theta - c)$.

Case 3: Otherwise. For any $\epsilon > 0$, we define two sets

$$A \triangleq \{(v', \theta') | v' \geq \theta', v' \in [v - \epsilon, v]\} \ ,$$
$$B \triangleq \{(v', \theta') | \theta' \geq v, v' \in [v - \epsilon, v]\} \ .$$

In this case, there exists an $\epsilon > 0$ such that regions A and B are both empty. In other words, for any value $v' \in (v - \epsilon, v)$, there exists $\theta(v') \in (v', v)$ such that $(v', g(v')) \in \text{supp}(G^*)$. Next, we define function $h(v') = -\mu^*(\theta(v'))$ for any $v' \in (v - \epsilon, v)$. For any $v_1, v_2 \in (v - \epsilon, v)$ with $v_1 < v_2$, it follows that

$$h(v_1) = \min_{\theta' \in [v_1, 1-c]} -\mu^*(\theta'), \ h(v_2) = \min_{\theta' \in [v_2, 1-c]} -\mu^*(\theta') \ .$$

Thus we know that $h(v_2) \geq h(v_1)$, which implies that the function h is weakly increasing over $(v - \epsilon, v)$. Next we are going to prove that the function h is also continuous over $(v - \epsilon, v)$.

First, we show the function h is right-continuous over $(v - \epsilon, v)$. For any $w \in (v', g(v'))$, it holds that $h(w) = h(v')$. Since the prior has positive density everywhere, we know that the function h is right-continuous over $(v - \epsilon, v)$.

Then, we show the function h is left-continuous over $(v-\epsilon, v)$. We prove it through contradiction. We assume that the function h is not left-continuous everywhere over $(v - \epsilon, v)$. For any tuple $(v', g(v')) \in \text{supp}(G^*)$,we define sets

$$R \triangleq \{(v'', \theta'') | \theta'' \in (v', v), v'' \in [0, v')\} \ ,$$
$$S \triangleq \{(v'', \theta'') | v'' \in (v - \epsilon, v'), \theta'' \in (v', v)\} \ .$$

Under this case, the region R must be empty. Besides, for any value $v'' \in (v - \epsilon, v')$, if $(v'', \theta'') \in \text{supp}(G^*)$, then it must hold that $(v'', \theta'') \in S$. Since h is increasing, we know that $\lim_{w \to v'^-} h(w) = \hat{h}$. Since $\lim_{w \to v'^-} \lambda^*(w) = \lambda^*(v')$, we know that $u(v') - c\hat{h} = u(v') - ch(v')$. Finally, we achieve that $\hat{h} = h(v')$, which forms a contradiction with the assumption. Till now, we have proved that the function h is continuous over $(v - \epsilon, v)$.

Next we aim to prove that h is flat over $(v - \epsilon, v)$. We assume that the image of the function h includes some open interval (y_1, y_2), thus the image is uncountable. Then, we define a set

$$T(y) \triangleq \{x | h(x) = y\} \ .$$

It is obvious that $T(y)$ is non-empty, and we let $z \in T(y)$. We know that $T(y)$ includes $(z, g(z))$. Therefore, we know that for any given y, its corresponding set $T(y)$ will consist of mutually disjoint intervals $(z, g(z))$. If we arbitrarily select a rational number from each such interval, then we would have obtained uncountably many rational numbers within the interval $(v-\epsilon, v)$. However, we know that the number of rational numbers within any interval is countable. This leads to a contradiction.

To sum up, we have proved that, under this case, there exists an $\delta > 0$, such that $-\mu^*(\theta') = u'(\theta)$ for any $\theta' \in (\theta - \delta, \theta]$. Thus, we directly have that

$$\lambda^*(v) = u'(\theta)q(v, \theta) + p(v, \theta) \ .$$

■

Lemma 15 (Case 3.3 in Algorithm 1)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, if there exists (v, θ) such that $v \in [\underline{v}, 1]$, $v \neq \theta + c$, and $v > v' = \inf \operatorname{supp}(G_{\cdot|\theta})$, then we have*

$$\lambda^*(v) = \frac{p(v,\theta)q(v',\theta) - p(v',\theta)q(v,\theta) + \lambda^*(v')q(v,\theta)}{q(v',\theta)} \ .$$

Proof of Lemma 15. Since $(v, \theta) \in \operatorname{supp}(G)$, by Theorem 2, we know that $\lambda^*(v) = -\mu^*(\theta)q(v, \theta) + p(v, \theta)$. Since the functions λ^*, p, and q are all continuous (Lemma 7), it holds that $\lambda^*(v') = -\mu^*(\theta)q(v', \theta) + p(v', \theta)$. By combining these two equations, we have

$$\lambda^*(v) = \frac{p(v,\theta)q(v',\theta) - p(v',\theta)q(v,\theta) + \lambda^*(v')q(v,\theta)}{q(v',\theta)} \ .$$

■

Lemma 16 (Algorithm 2)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any $\theta \in \operatorname{supp}(G^*_\theta)$ and any two distinct $v_1, v_2 \neq \theta + c$ such that $(v_1, \theta), (v_2, \theta) \in \operatorname{supp}(G^*)$, we have that*

$$\mu^*(\theta) = \frac{p(v_1,\theta) - \lambda^*(v_1)}{q(v_1,\theta)} = \frac{p(v_2,\theta) - \lambda^*(v_2)}{q(v_2,\theta)} \ .$$

Proof of Lemma 16. It holds directly by Theorem 2. ■

Lemma 17 (Algorithm 2)**.** *Given G^* and (λ^*, μ^*) are both optimal to the primal and dual problems, for any $\theta \in [-c, 1-c]$, it holds that*

$$\max_{v \in (\theta+c, 1]} \frac{u(\theta) - \lambda^*(v)}{v - \theta - c} \leq \mu^*(\theta) \leq \min\left\{\min_{v \in [0,\theta)} \frac{\lambda^*(v) - u(v)}{c}, \min_{v \in [\theta, \theta+c)} \frac{u(\theta) - \lambda^*(v)}{v - \theta - c}\right\} \ .$$

Proof of Lemma 17. By Constraint (6) and Theorem 2, we directly have that

$$\max_{v \in (\theta+c, 1]} \frac{u(\theta) - \lambda^*(v)}{v - \theta - c} \leq \mu^*(\theta) \leq \min\left\{\min_{v \in [0,\theta)} \frac{\lambda^*(v) - u(v)}{c}, \min_{v \in [\theta, \theta+c)} \frac{u(\theta) - \lambda^*(v)}{v - \theta - c}\right\} \ .$$

■

With the above lemmas, finally, we can provide the whole proof of Theorem 3.

Proof of Theorem 3. This theorem directly holds by Lemmas 3 and 8 to 17. ■

C Supplementary Materials in Section 4

C.1 An Example of Equilibrium Non-Existence

First, we provide an example below to show that the equilibrium may fail to exist when there are masses in the senders' priors.

Example 2. *Consider a two-sender game, where both boxes share a common inspection cost* $c = 0.1$*. Sender* 1*'s prior puts a mass of size* 0.5 *at* 0.4 *and* 0.6*, while sender* 2*'s prior puts a mass of size* 1 *at* 0.5*. Sender* 2*'s strategy space degenerates to a single point, that is the prior itself with the amortized value fixed to* 0.4*. Faced with this, sender* 1 *has no best response. To see this, consider any 2-D distribution* G_1 *of sender* 1*. Sender* 1 *gets positive utility only when the amortized value sampled from* G_1 *is at least* 0.4*. Let* $p \triangleq \mathbf{Pr}_{(v_1,\theta_1)\sim G_1}[v_1 = 0.6, \theta_1 > 0.4]$ *with* $p < 0.5$*, the utility of this value-index pair is* 1*. Let* $q \triangleq \mathbf{Pr}_{(v_1,\theta_1)\sim G_1}[v_1 = 0.4, \theta_1 > 0.4]$ *with* $q < 0.5$*, the utility of this pair is some* $\alpha \in (0, 1)$ *under certain non-degenerate tie-breaking rule. Let* $r \triangleq \mathbf{Pr}_{(v_1,\theta_1)\sim G_1}[\theta_1 = 0.4]$*, the utility of this case is some* $\beta \in (0, \alpha)$*. The sender's expected utility with* G_1 *is* $p + q \cdot \alpha + r \cdot \beta$*, which does not attain a maximum for any feasible* G_1 *as the supremum is achieved only when* $p + q = 1$*.*

C.2 Proof of Theorem 4

We proceed with the proof in the following three steps. Briefly speaking, in Step 1, we construct a finite game called m-discrete approximation game for each granularity $m \in \mathbb{Z}^+$, and directly apply Nash's Theorem to show the existence of a discrete equilibrium. In Step 2, we show that the sequence of discrete equilibria has a subsequence that weakly converges to some distribution profile, where any of the distributions forms a feasible action for each sender. In Step 3, we prove that the limit profile indeed constitutes an equilibrium of the original game through establishing the convergence of utilities along with the convergence of equilibrium strategies. For ease of presentation, we assume all senders share a common inspection cost c, that is $c_1 = \cdots = c_N = c$.

Step 1: Constructing m-discrete approximation games. By Lemma 2, each sender i's strategy space $\mathcal{G}(F_i, c)$ is a compact and convex set containing all feasible 2-D distributions. By Krein-Milman Theorem, any convex and compact set is the closed convex hull of its extreme points; moreover, each element within the set can be represented as a convex combination of its extreme points. Thus, each sender's strategy space can be precisely characterized by its extreme points. Recall from Definition 4 that the space $\mathcal{G}(F_i, c)$ is subject to an infinite number of constraints; the convex space $\mathcal{G}(F_i, c)$ may therefore have an infinite number of extreme points. Therefore, even if one takes the set of these extreme points as the action set, Nash's theorem does not apply. To address this problem, we construct finite games that not only discretize the support of the value and the index, but also modify constraints in Definition 4. Each finite game is parameterized by an integer $m \in \mathbb{Z}^+$.

Definition 6 (m-Discrete Approximation Game)**.** *Fix any* $m \in \mathbb{Z}^+$*, let* $V^m \triangleq \{0, \frac{1}{2^m}, \cdots, 1\}$ *and* $\Theta^m \triangleq \{-c, \frac{1}{2^m} - c, \ldots, 1 - c\}$ *be the discretized support for the value and the index, respectively. The* m-discrete approximation game *is as follows:*

- ***Strategy space:*** *For each sender* i*, the strategy space* $\mathcal{S}_i^m$ *is a subset of distributions on*

$V^m \times \Theta^m$. *Each* $\mathbf{p} \in \mathcal{S}_i^m$*, with* $p_{i,j}$ *denoting the probability on the point* $(\frac{i}{2^m}, \frac{j}{2^m} - c)$*, satisfies*

$$\sum_{i=0}^{2^m} p_{i,j} \cdot \left[\frac{i}{2^m} - \frac{j}{2^m} + c\right]_+ = c \cdot \sum_{i=0}^{2^m} p_{i,j} \ , \quad \forall j \in \{0\} \cup [2^m] \ ; \tag{21}$$

$$\sum_{j=0}^{2^m} p_{i,j} = F_i\left(\frac{i}{2^m}\right) - F_i\left(\left[\frac{i-1}{2^m}\right]_+\right) \ , \quad \forall i \in \{0\} \cup [2^m] \ . \tag{22}$$

- ***Action space:*** *Each sender* i*'s action space, denoted by* $\mathcal{A}_i^m$*, is defined as the set of all extreme points of* $\mathcal{S}_i^m$.
- ***Utility:*** *Given a strategy profile* $(G_1^m, ..., G_N^m)$ *where each* $G_i^m \in \mathcal{A}_i^m$*, the receiver receives* $(v_i^m, \theta_i^m) \sim G_i^m$ *for each* i*, and chooses a sender whose amortized value* $\kappa_i^m \triangleq \min(v_i^m, \theta_i^m)$ *is non-negative and the largest. If all amortized values are negative, the receiver chooses no one. If there is a tie among multiple senders, break the tie uniformly at random.*[7] *A sender gets utility* 1 *if chosen by the receiver, and* 0 *otherwise.*

Lemma 18. *For any* $m \in \mathbb{Z}^+$*, each sender* i*'s strategy space* $\mathcal{S}_i^m$ *is non-empty, convex, and compact. Each sender* i*'s action space* $\mathcal{A}_i^m$ *is finite.*

Proof of Lemma 18. First, we prove that the space $\mathcal{S}_i^m$ is non-empty. For any $m \in \mathbb{Z}^+$ and each sender i, the full-revelation strategy belongs to the space $\mathcal{S}_i^m$, which makes the space $\mathcal{S}_i^m$ non-empty.

Second, we prove the space $\mathcal{S}_i^m$ is convex. For any $m \in \mathbb{Z}^+$, consider any pair of feasible strategies G^m, $H^m \in \mathcal{S}_i^m$ (with probability mass $g_{i,j}$ and $h_{i,j}$ on the point $(\frac{i}{2^m}, \frac{j}{2^m} - c)$ respectively) and any $\lambda \in [0, 1]$, we construct the convex combination T^m where each mass $t_{i,j} = \lambda \cdot g_{i,j} + (1 - \lambda) \cdot h_{i,j}$. It is obvious that for each $j \in \{0\} \cup [2^m]$,

$$\begin{aligned}
&\sum_{i=0}^{2^m} t_{i,j} \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} \\
=\,&\lambda \cdot \sum_{i=0}^{2^m} g_{i,j} \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} + (1-\lambda) \cdot \sum_{i=0}^{2^m} h_{i,j} \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} \\
=\,&\lambda \cdot c \cdot \sum_{i=0}^{2^m} g_{i,j} + (1-\lambda) \cdot c \cdot \sum_{i=0}^{2^m} h_{i,j} \ = \ c \cdot \sum_{i=0}^{2^m} t_{i,j} \ ,
\end{aligned}$$

and for each $i \in \{0\} \cup [2^m]$,

$$\sum_{i=0}^{2^m} t_{i,j} = \lambda \cdot \sum_{j=0}^{2^m} g_{i,j} + (1-\lambda) \cdot \sum_{j=0}^{2^m} h_{i,j} = F\left(\frac{i}{2^m}\right) - F\left(\max\left\{0, \frac{i-1}{2^m}\right\}\right) \ .$$

So we know that the convex combination T^m also belongs to the space $\mathcal{S}_i^m$, which makes the space $\mathcal{S}_i^m$ convex.

Third, we prove that the space $\mathcal{S}_i^m$ is compact. Since the space $\mathcal{S}_i^m$ is indeed a measure space

[7] Here we adopt the uniformly random tie-breaking rule for convenience of presentation. As we remarked above, the proof goes through for a host of other tie-breaking rules.

over $V^m \times \Theta^m$, the space $\mathcal{S}_i^m$ is bounded. For any $m \in \mathbb{Z}^+$ and any sender i, we assume that sequence $\{G^k\}_{k\in\mathbb{Z}^+}$ converges to some discrete distribution G where each $G^k \in \mathcal{S}_i^m$. So the sequence $\{p_{i,j}^k\}_{k\in\mathbb{Z}^+}$ converges to $p_{i,j}$ where $p_{i,j}^k$ denotes the probability measure of strategy G^k on the point $(\frac{i}{2^m}, \frac{j}{2^m} - c)$ for any $i, j \in \{0\} \cup [2^m]$, and $p_{i,j}$ denotes the probability measure of strategy G on the same point. This implies that there exists $K > 0$ such that for any $k > K$ and any $\epsilon > 0$, we have $|p_{i,j}^k - p_{i,j}| < \epsilon$ for any $i, j \in \{0\} \cup [2^m]$. If we assume that there exist some m and $j \in \{0\} \cup [2^m]$ such that $\sum_{i=0}^{2^m} p_{i,j} \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} = d \cdot \sum_{i=0}^{2^m} p_{i,j} \neq c \cdot \sum_{i=0}^{2^m} p_{i,j}$. There exists $K_1 > 0$ such that for any $k > K_1$, we have

$$\left| \sum_{i=0}^{2^m} p_{i,j}^k \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} - \sum_{i=0}^{2^m} p_{i,j} \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} \right| < \frac{1}{3}\left| d \cdot \sum_{i=0}^{2^m} p_{i,j} - c \cdot \sum_{i=0}^{2^m} p_{i,j} \right|.$$

There exists $K_2 > 0$ such that for any $k > K_2$, we have

$$\left| c \cdot \sum_{i=0}^{2^m} p_{i,j}^k - c \cdot \sum_{i=0}^{2^m} p_{i,j} \right| < \frac{1}{3}\left| d \cdot \sum_{i=0}^{2^m} p_{i,j} - c \cdot \sum_{i=0}^{2^m} p_{i,j} \right|.$$

Thus, we know for any $k > \max\{K_1, K_1\}$, we have

$$\left| \sum_{i=0}^{2^m} p_{i,j}^k \cdot \max\left\{\frac{i}{2^m} - \frac{j}{2^m} + c, 0\right\} - c \cdot \sum_{i=0}^{2^m} p_{i,j}^k \right| > \frac{1}{3}\left| d \cdot \sum_{i=0}^{2^m} p_{i,j} - c \cdot \sum_{i=0}^{2^m} p_{i,j} \right|,$$

which forms a contradiction with the fact that $G^k \in \mathcal{S}_i^m$. Furthermore, if we assume that there exist some m and $j \in \{0\} \cup [2^m]$ such that $\sum_{j=0}^{2^m} p_{i,j} \neq F\left(\frac{i}{2^m}\right) - F\left(\max\left\{0, \frac{i-1}{2^m}\right\}\right)$, then we can achieve a contradiction in the same manner. So we have proved that the space $\mathcal{S}_i^m$ is closed. Then we can directly make space $\mathcal{G}(F, c)$ compact through the Heine-Borel Theorem.

Last, we prove that space $\mathcal{A}_i^m$ is finite. By Definition 6, set $\mathcal{S}_i^m$ is formed by $O(2^m)$ linear constraints, which implies that there are a finite number of extreme points of set $\mathcal{S}_i^m$. Thus, the action space $\mathcal{A}_i^m$ is finite. ■

Step 2: Showing the limit of equilibrium strategies is feasible. By Lemma 18, each m-discrete approximation game is finite, there being a finite number of senders, and each sender i having a finite action space. Thus, Nash's Theorem applies, and there is an equilibrium $(\tilde{G}_1^m, ..., \tilde{G}_N^m)$, where for each sender i, $\tilde{G}_i^m \in \mathcal{S}_i^m$ is a mixed equilibrium strategy. For each sender i, these equilibrium strategies form a sequence of 2-D distributions: $\{\tilde{G}_i^m\}_{m\in\mathbb{Z}^+}$. We want to establish the convergence of this sequence, so we first need to introduce the following Helly's Selection Theorem.

Lemma 19 (Helly's Selection Theorem)**.** *Let $\{G^m\}_{m\in\mathbb{Z}^+}$ be a sequence of CDFs which is tight,*[8] *then there exists a subsequence $\{m(k)\}_{k\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+$ such that $\{G^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF G.*[9]

By Helly's Selection Theorem, there is a subsequence that weakly converges to a 2-D distribution; we further show that this limit is a feasible for each sender i.

[8]Take the one-dimensional case as an example, a sequence of functions $\{G^m\}_{m\in\mathbb{Z}^+}$ is tight, if and only if $\forall\epsilon > 0$ there exists an interval $[a, b]$ such that for each $m \in \mathbb{Z}^+$ we have $G^m(b) - G^m(a) > 1 - \epsilon$.

[9]Take the one-dimensional case as an example, sequence $\{G^m\}_{m\in\mathbb{Z}^+}$ weakly converges to CDF G if and only if $\lim_{m\to\infty} G^m(x) = G(x)$ for each point x at which G is continuous.

Lemma 20. *There exists a subsequence $\{m(k)\}_{k\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+$ such that the sequence $\{(\tilde{G}_i^{m(k)})\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain 2-D distribution $\tilde{G}_i$. Furthermore, $\tilde{G}_i$ is a feasible strategy for sender i in the original game.*

Proof of Lemma 20. For any $m \in \mathbb{Z}^+$, space $\mathcal{S}_i^m$ is a measure space over $[0,1]\times[-c,1-c]$, which makes space $\mathcal{S}_i^m$ tight. By Lemma 19, for each sender $i \in [N]$, there exists a subsequence $\{m_i(k)\}_{k\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+$ such that $\{\tilde{G}_i^{m_i(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain 2-d CDF $\tilde{G}_i$. This means that, for any continuous point $(\hat{v},\hat{\theta})$ of distribution $\tilde{G}_i$, the sequence $\{\tilde{G}_i^{m_i(k)}(\hat{v},\hat{\theta})\}_{k\in\mathbb{Z}^+}$ converges to $\tilde{G}_i(\hat{v},\hat{\theta})$. Next, it can be shown that, there exists a common subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ such that for each sender i, the sequence $\{\tilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF $\tilde{G}_i$. To see this, we first find a subsequence $\{m_1(k)\}_{k\in\mathbb{Z}^+}$ of $\mathbb{Z}^+$ such that $\{\tilde{G}_1^{m_1(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to $\tilde{G}_1$. We then find a subsequence $\{m_2(k)\}_{k\in\mathbb{Z}^+}$ of $\{m_1(k)\}_{k\in\mathbb{Z}^+}$ such that $\{\tilde{G}_2^{m_2(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to $\tilde{G}_2$. It is known that if a sequence converges, any subsequence of it also converges to the same limit. Hence, $\{\tilde{G}_1^{m_2(k)}\}_{k\in\mathbb{Z}^+}$ also weakly converges to $\tilde{G}_1$. In the same manner, we obtain a common subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ of $\mathbb{Z}^+$ such that for each sender i, the sequence $\{\tilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF $\tilde{G}_i$.

Then, we are going to show that the limit 2-d distribution $\tilde{G}_i$ is a feasible strategy of sender i in the original game. For convenience, for each sender i, we let $\{\tilde{G}^k\}_{k\in\mathbb{Z}^+}$ denote the sequence of discrete equilibrium strategies, and let $\tilde{G}$ denote the limit. We divide this proof into two steps.

First, we prove that $\mathbb{E}_{v\sim\tilde{G}_{\cdot|\theta}}[\max\{v-\theta,0\}] = c$ **for any** $\theta \in [-c,1-c]$**.** Let's consider any index $\theta \in (-c,1-c)$ since the cases of index $-c$ and index $1-c$ are trivial. For any $\hat{\theta} \in \mathrm{supp}(\tilde{G}_\theta)$, we assume that

$$\int_{v=0}^{1} |c-\max\{v-\theta,0\}|\ \tilde{g}(v,\hat{\theta})\ \mathrm{d}v = d > 0\ .$$

By the weak convergence of sequence $\{\tilde{G}^k(\hat{v},\hat{\theta})\}_{k\in\mathbb{Z}^+}$ to distribution $\tilde{G}$, there exists $K_1 > 0$ such that when $k > K_1$, we have

$$\left|\int_{v=0}^{1} |c-\max\{v-\theta,0\}|\ \tilde{g}(v,\hat{\theta})\ \mathrm{d}v - \int_{v=0}^{1} |c-\max\{v-\theta,0\}|\ \tilde{g}^k(v,\hat{\theta})\ \mathrm{d}v\right| < \frac{d}{2}\ .$$

This implies that for any $k > K_1$, we have

$$\int_{v=0}^{1} |c-\max\{v-\theta,0\}|\ \tilde{g}^k(v,\hat{\theta})\ \mathrm{d}v \in \left(\frac{d}{2},\frac{3d}{2}\right)\ ,$$

which forms a contradiction with the fact $\int_{v=0}^{1} |c-\max\{v-\theta,0\}|\ \tilde{g}^k(v,\hat{\theta})\ \mathrm{d}v = 0$, and makes this assumption invalid.

Second, we prove that $\int_{\theta=-c}^{1-c}\ \mathrm{d}\tilde{G}(v,\theta) = f(v)$ **for any** $v \in [0,1]$**.** Let's consider any value $v \in (0,1]$, since the case of value 0 is trivial. For any $\hat{v} = \alpha\cdot 2^{-\beta}$ for some $\beta \in \mathbb{Z}^+$ and some $\alpha \in [2^\beta]$, then for any $k > \beta$, there exists $t \in \{0\}\cup[2^k]$ such that $\hat{v} = \frac{t}{2^k}$ and $G^k(\frac{t}{2^k},1-c) - G^k(\frac{t-1}{2^k},1-c) = F(\frac{t}{2^k}) - F(\frac{t-1}{2^k})$. Dividing both sides of the equation by 2^{-k}, we get

$$\frac{G^k(\frac{t}{2^k},1-c) - G^k(\frac{t-1}{2^k},1-c)}{2^{-k}} = \frac{F(\frac{t}{2^k}) - F(\frac{t-1}{2^k})}{2^{-k}}\ .$$

When $k \to \infty$, we know that $\int_{\theta=-c}^{1-c} \mathrm{d}G(\hat{v}, \theta) = f(\hat{v})$. For any $\hat{v} \neq \alpha \cdot 2^{-\beta}$ for any $\beta \in \mathbb{Z}^+$ and any $\alpha \in [2^\beta]$, then for any $k \in \mathbb{Z}^+$ there exist $t \in [2^m]$ such that $\hat{v} \in (\frac{t-1}{2^k}, \frac{t}{2^k})$ and $G^k(\frac{t}{2^k}, 1-c) - G^k(\hat{v}, 1-c) = F(\frac{t}{2^k}) - F(\frac{t-1}{2^k})$. Dividing both sides of the equation by 2^{-k}, we get

$$\frac{G^k(\frac{t}{2^k}, 1-c) - G^k(\hat{v}, 1-c)}{2^{-k}} = \frac{F(\frac{t}{2^k}) - F(\frac{t-1}{2^k})}{2^{-k}} .$$

When $k \to \infty$, we know that $\int_{-c}^{1-c} \mathrm{d}G(\hat{v}, \theta) = f(\hat{v})$.

Combining these two steps, we have proved that each sender i's strategy in the limit, $\tilde{G}_i$, is indeed a feasible strategy for the original game. ■

Step 3: Proving the limit strategy profile $(\tilde{G}_1, ..., \tilde{G}_N)$ is an equilibrium in the original game. To this end, we need to show that each sender's utility converges in the subsequence along with the convergence of discrete equilibrium strategies. An obstacle to this is that the senders' utilities may be discontinuous in their strategies if there are ties in $(\tilde{G}_1, ..., \tilde{G}_N)$ with strictly positive probability. To rule out this case, we first prove that, in the subsequence of approximation games, the probability of ties in any neighborhood diminishes. We only need to show this for amortized values that actually affect the senders' utilities (above the smallest winning amortized value defined below).

Definition 7 (Smallest Winning Amortized Value)**.** *For any profile of amortized value distributions $(K_1, ..., K_N)$, the* smallest winning amortized value *is $\underline{\kappa} \triangleq \max_{i \in [N]} \inf \operatorname{supp}(K_i)$.*

Each sender has zero utility for realizing an amortized value below the smallest winning amortized value $\underline{\kappa}$. If sender i's strategy has a mass in K_i below $\underline{\kappa}$, it does not cause discontinuity in anyone's utility. Therefore we need only to focus on the part of the distribution at or above $\underline{\kappa}$. For the limit strategy profile $(\tilde{G}_1, ..., \tilde{G}_N)$, denote each sender i's distribution of amortized value as $\tilde{K}_i$, and the smallest winning amortized value as $\underline{\kappa}$.

Lemma 21. *There exist no amortized value $\hat{\kappa} \in [\underline{\kappa}, 1-c]$ and two distinct senders $i, j \in [N]$ such that $\tilde{K}_i, \tilde{K}_j$ both assign positive probabilities at $\hat{\kappa}$.*

Proof of Lemma 21. By the assumption of the prior, there exists no tie at index $1-c$ in any feasible strategy profile. We assume that there exists an index $\hat{\theta} \in [\underline{\theta}, 1-c)$ such that sender 1 and 2's strategies, $\tilde{G}_1, \tilde{G}_2$, simultaneously assign a positive mass at index $\hat{\theta}$. Let $\phi_1(\hat{\theta}) - \phi_1(\hat{\theta}^-) = p_1 > 0$ and $\phi_2(\hat{\theta}) - \phi_2(\hat{\theta}^-) = p_2 > 0$. We want to prove that discrete strategies, $\tilde{G}_1^m$ and $\tilde{G}_2^m$, also assign a big enough probability at index $\hat{\theta}$ for sufficiently large m, which violates the equilibrium conditions. By Lemma 20, strategy $\tilde{G}_1$ must be a multi-value row at row $\hat{\theta}$, that is $\inf \operatorname{supp}(\tilde{G}_{1,\cdot|\hat{\theta}}) < \sup \operatorname{supp}(\tilde{G}_{1,\cdot|\hat{\theta}})$. In the same manner, we have $\inf \operatorname{supp}(\tilde{G}_{2,\cdot|\hat{\theta}}) < \sup \operatorname{supp}(\tilde{G}_{2,\cdot|\hat{\theta}})$. First, based on the definition of u, for each sender $i \in [N]$, we define the discrete version of the compressed interim utility as

$$\phi_i^m(x) = \prod_{j \neq i} \left(\tilde{G}_j^m(1, x) + \tilde{G}_j^m(x, 1-c) - \tilde{G}_j^m(x, x) \right) \ \forall x \in [-c, 1-c].$$

Based on whether the index $\hat{\theta}$ belongs to the discrete support of index Θ^m for some $m \in \mathbb{Z}^+$, we divide this part of proof into two cases.

Case 1: There exist $\beta \in \mathbb{Z}^+$ and $\alpha \in [2^\beta]$ such that $\hat{\theta} = \alpha \cdot 2^{-\beta} - c$. We take sender 1 as an example, and the proof of sender 2 follows the same idea. There exists $m_1 > 0$ such that when

$m > m_1$, $(\hat{\theta} - 8 \cdot 2^{-m}, \hat{\theta} + 8 \cdot 2^{-m}) \subset (\underline{\theta}, 1-c)$ and spreading the row $\hat{\theta}$ of the discrete equilibrium strategy to any row $\theta' \in (\hat{\theta}, \hat{\theta} + 8 \cdot 2^{-m})$ and any row $\theta'' \in (\hat{\theta} - 8 \cdot 2^{-m}, \hat{\theta})$ is possible, and will not cause any value of row θ exceed the line $v = \theta$. We define $\epsilon_1 = \frac{p_1}{100}$, then there exists $m_2 > 0$ such that when $m > m_2$, it holds that $|\phi_1^m(\hat{\theta}^+) - \phi_1^m(\hat{\theta}^-) - p_1| < \epsilon_1$. There exists $m_3 > 0$ such that when $m > m_3$, it holds that $\hat{\theta} - 7 \cdot 2^{-m} > \underline{\theta}$ and $\phi_1^m(\hat{\theta}^-) - \phi_1^m(\hat{\theta} - 7 \cdot 2^{-m}) < \epsilon_2$ where $\epsilon_2 = \frac{1}{100}p_1$. There exists $m_4 > 0$ such that when $m > m_4$, it holds that $\hat{\theta} + 2^{-m} < 1-c$ and $\phi_1^m(\hat{\theta} + 2^{-m}) - \phi_1^m(\hat{\theta}^+) < \epsilon_3$ where $\epsilon_3 = \frac{1}{100}p_1$. Based on these inequalities, we achieve that $3\epsilon_1 + \epsilon_2 < 3p_1$, which implies that

$$\frac{7}{8}(p_1 - \epsilon_1 + \epsilon_2 + \epsilon_3) > \epsilon_2 + \frac{p_1 - \epsilon_1}{2} \ ,$$

$$\frac{7}{8}(p_1 + \epsilon_1 + \epsilon_2 + \epsilon_3) > \epsilon_2 + \frac{p_1 + \epsilon_1}{2} \ .$$

This implies that, for any $m > \max\{m_1, m_2, m_3, m_4\}$, spreading the row $\hat{\theta}$ of discrete equilibrium strategy G_1^m to row $\hat{\theta} - 7 \cdot 2^{-m}$ and row $\hat{\theta} + 2^{-m}$ can bring a strictly positive utility increase, which violates the equilibrium conditions.

Case 2: $\hat{\theta} \neq \alpha \cdot 2^{-\beta} - c$ for any $\beta \in \mathbb{Z}^+$ and any $\alpha \in [2^\beta]$. We take sender 1 as an example, and the proof of sender 2 follows the same idea. For any $m \in \mathbb{Z}^+$, there exists $k \in \{0\} \cup [2^m]$ such that $\hat{\theta} \in (\frac{k}{2^m}, \frac{k+1}{2^m})$. There exists $m_1 > 0$ such that when $m > m_1$, it holds that $(k \cdot 2^{-m} - 8 \cdot 2^{-m}, (k+1) \cdot 2^{-m} + 8 \cdot 2^{-m}) \subset (\underline{\theta}, 1-c)$, and spreading the row $k \cdot 2^{-m}$ of the discrete equilibrium strategy to any row $\theta' \in ((k+1) \cdot 2^{-m}, (k+1) \cdot 2^{-m} + 8 \cdot 2^{-m})$ and any row $\theta'' \in (k \cdot 2^{-m} - 8 \cdot 2^{-m}, k \cdot 2^{-m})$ is possible, and will not cause any value of row $k \cdot 2^{-m}$ exceed the line $v = \theta$. There exists $m_2 > 0$ such that when $m > m_2$, it holds that $|\phi_1^m(((k+1) \cdot 2^{-m})^+) - \phi_1^m((k \cdot 2^{-m})^-) - p_1| < \epsilon_1$ where $\epsilon_1 = \frac{p_1}{100}$. There exists $m_3 > 0$ such that when $m > m_3$, it holds that $k \cdot 2^{-m} - 5 \cdot 2^{-m} > \underline{\theta}$ and $\phi_1^m((k \cdot 2^{-m})^-) - \phi_1^m(k \cdot 2^{-m} - 5 \cdot 2^{-m}) < \epsilon_2$ where $\epsilon_2 = \frac{p_1}{100}$. There exists $m_4 > 0$ such that when $m > m_4$, it holds that $(k+1) \cdot 2^{-m} + 2^{-m} < 1-c$ and $\phi_1^m((k+1) \cdot 2^{-m} + 2^{-m}) - \phi_1^m(((k+1) \cdot 2^{-m})^+) < \epsilon_3$ where $\epsilon_3 = \frac{p_1}{100}$. Based on these inequalities, we have $\epsilon_1 + 3\epsilon_2 < p_1$, which implies that

$$\frac{5}{8}(p_1 - \epsilon_1 + \epsilon_2 + \epsilon_3) > \epsilon_2 + \frac{p_1 - \epsilon_1}{2} \ , \quad \frac{5}{8}(p_1 + \epsilon_1 + \epsilon_2 + \epsilon_3) > \epsilon_2 + \frac{p_1 + \epsilon_1}{2} \ .$$

This implies that, for any $m > \max\{m_1, m_2, m_3, m_4\}$, spreading the row $k \cdot 2^{-m}$ of discrete equilibrium strategy G_1^m to row $k \cdot 2^{-m} - 5 \cdot 2^{-m}$ and row $(k+1) \cdot 2^{-m} + 2^{-m}$ can bring a strictly positive utility increase, which violates the equilibrium conditions.

In summary, given the assumption, the converging discrete strategy profile cannot form an equilibrium, which make this assumption invalid. Thus, we have proved that there is no tie in the limit strategy profile $(\tilde{G}_1, ..., \tilde{G}_N)$ at and above $\underline{\theta}$. ■

With the obstacle of discontinuity cleared, we obtain the convergence of utility functions along with the convergence of discrete equilibrium strategies. Here we comes the final step of the proof.

Lemma 22. *The limit strategy profile $(\tilde{G}_1, ..., \tilde{G}_N)$ is indeed an equilibrium in the original game.*

Proof of Lemma 22. Consider each sender $i \in [N]$ and any feasible strategy $G_i \in \mathcal{G}_i(F_i, c)$, we want to show that strategy $\tilde{G}_i$ achieves a higher expected utility than strategy G_i, given others strategies $\tilde{G}_{-i}$. We divide this part of proof into two cases.

Case 1: There exist ties over $[\underline{\theta}, 1-c]$ in strategy profile $(G_i, \tilde{G}_{-i})$. We assume that there exists a tie at index $\hat{\theta}$ in strategy profile $(G_i, \tilde{G}_{-i})$. By the assumption of the prior, strategy

G_i must be a multi-value row at row $\hat{\theta}$, that is $\inf \operatorname{supp}(G_{i,\cdot|\hat{\theta}}) < \sup \operatorname{supp}(G_{i,\cdot|\hat{\theta}})$, which implies that strategy G_i can spread the probability of index $\hat{\theta}$ to index $\hat{\theta}+\epsilon$ and index $\hat{\theta}-\epsilon$ for sufficiently small $\epsilon > 0$. By the fact that $\phi_i(\hat{\theta}^-) < \phi_i(\hat{\theta}) < \phi_i(\hat{\theta}^+)$, we know spreading the probability of strategy G_i at index $\hat{\theta}$ to index $\hat{\theta}+\epsilon$ and index $\hat{\theta}-\epsilon$ achieves a utility increase for sufficiently small $\epsilon > 0$. Therefore, we know that, given others' strategies, the best response strategy of sender i will not form a tie with the strategies of others within the interval $[\underline{\theta}, 1-c]$. Thus, in Case 2, we only need to prove that for those strategies that do not form a tie with the others' strategies within $[\underline{\theta}, 1-c]$, strategy $\tilde{G}_i$ will achieve a higher expected utility. This implies that strategy $\tilde{G}_i$ is the best response to others' strategies.

Case 2: There is no tie over $[\underline{\theta}, 1-c]$ in strategy profile $(G_i, \tilde{G}_{-i})$. Let $\tilde{u}_i$ denote sender i's interim utility function given others' strategies $\tilde{G}_{-i}$. Let $\tilde{u}_i^m$ denote sender i's interim utility function given others' strategies $\tilde{G}_{-i}^m$. By Lemma 21, there is no tie over $[\underline{\theta}, 1-c]$ in the limit strategy profile $(\tilde{G}_1, \ldots, \tilde{G}_N)$. By the definition of weak convergence, we have that

$$\lim_{m\to\infty} \int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ \tilde{g}_i^m(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_{[0,1]\times[-c,1-c]} \tilde{u}_i(v,\theta)\ \tilde{g}_i(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ . \tag{23}$$

For each sender $i \in [N]$ and any feasible strategy $G_i \in \mathcal{G}_i(F_i, c)$ that do not form a tie with others' strategies within $[\underline{\theta}, 1-c]$, we construct a 2-dimensional CDFs sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ where each $G_i^m \in \mathcal{S}_i^m$, and the sequence weakly converges to strategy G_i. Specifically, for $\forall m \in \mathbb{Z}^+$, we define G_i^m as below.

$$G_i^m(v,\theta) = \begin{cases} G_i(v,\theta)\ , & \text{if } (v,\theta) \in V^m \times \Theta^m\ , \\ G_i(\max\{t : t \in V^m,\ t \le v\}, \max\{t : t \in \Theta^m,\ t \le \theta\})\ , & \text{otherwise}\ . \end{cases}$$

First, we show that the sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to strategy G_i. We consider any pair of $(v,\theta) \in [0,1] \times [-c, 1-c]$. If there exist $\alpha_1 \in \mathbb{Z}^+$ and $\beta_1 \in [2^{-\alpha_1}]$ such that $v = \beta_1 \cdot 2^{-\alpha_1}$ and $\alpha_2 \in \mathbb{Z}^+$ and $\beta_2 \in [2^{-\alpha_2}]$ such that $\theta = \beta_2 \cdot 2^{-\alpha_2}$, then when $m > \max\{\alpha_1, \alpha_2\}$, we have $G_i^m(v,\theta) = G_i(v,\theta)$ which implies that sequence $\{G_i^m(v,\theta)\}_{m\in\mathbb{Z}^+}$ converges to $G_i(v,\theta)$. If $v \ne \beta_1 \cdot 2^{-\alpha_1}$ for any $\alpha_1 \in \mathbb{Z}^+,\ \beta_1 \in [2^{-\alpha_1}]$ or $\theta \ne \beta_2 \cdot 2^{-\alpha_2}$ for any $\alpha_2 \in \mathbb{Z}^+,\ \beta_2 \in [2^{-\alpha_2}]$, then for any $m \in \mathbb{Z}^+$, we have $G_i^m(v,\theta) = G_i(\max\{t : t \in V^m,\ t \le v\}, \max\{t : t \in \Theta^m,\ t \le \theta\})$. When m goes to infinity, sequence $\{\max\{t : t \in V^m,\ t \le v\}\}_{m\in\mathbb{Z}^+}$ converges to v, and sequence $\{\max\{t : t \in \Theta^m,\ t \le \theta\}\}_{m\in\mathbb{Z}^+}$ converges to θ. So we have sequence $\{G_i^m(v,\theta)\}_{m\in\mathbb{Z}^+}$ also converges to $G_i(v,\theta)$. By the fact that sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to strategy G_i, using a similar method of Lemma 20, we can also prove that there exists $m_1 > 0$ such that when $m < m_1$, $G_i^m \in \mathcal{S}_i^m$.

By Equation (23), we have

$$
\begin{aligned}
&\int_{[0,1]\times[-c,1-c]} \tilde{u}_i(v,\theta)\ g_i(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} \tilde{u}_i(v,\theta)\ \tilde{g}_i(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta \\
&= \lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} \tilde{u}_i(v,\theta)\ g_i(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ g_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta\right) \\
&+ \lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ g_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ \tilde{g}_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta\right) \\
&= \lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} G_i^m(v,\theta)\ \mathrm{d}\tilde{u}_i^m(v,\theta) - \int_{[0,1]\times[-c,1-c]} G_i(v,\theta)\ \mathrm{d}\tilde{u}_i(v,\theta)\right) \\
&+ \lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ g_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} \tilde{u}_i^m(v,\theta)\ \tilde{g}_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta\right) .
\end{aligned}
$$

Combining the facts that sequence $\{(\tilde{G}_1^m, ..., \tilde{G}_N^m)\}_{m\in\mathbb{Z}^+}$ weakly converges to $(\tilde{G}_1, ..., \tilde{G}_N)$, sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to G_i, and there is no tie over $[\underline{\theta}, 1-c]$ in strategy profile $(G_i, \tilde{G}_{-i})$, we have that

$$
\lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} G_i(v,\theta)\ \mathrm{d}u_i(v,\theta) - \int_{[0,1]\times[-c,1-c]} G_i^m(v,\theta)\ \mathrm{d}u_i^m(v,\theta)\right) = 0\ . \tag{24}
$$

In addition to the fact $(\tilde{G}_1^m, \ldots, \tilde{G}_N^m)$ is an equilibrium in the m-th discrete approximation game, we have

$$
\lim_{m\to\infty}\left(\int_{[0,1]\times[-c,1-c]} u_i^m(v,\theta)\ g_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} u_i^m(v,\theta)\ \tilde{g}_i^m(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta\right) \le 0\ . \tag{25}
$$

Combining Inequalities (24) and (25), we have

$$
\int_{[0,1]\times[-c,1-c]} u_i(v,\theta)\ g_i(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta - \int_{[0,1]\times[-c,1-c]} u_i(v,\theta)\ \tilde{g}_i(v,\theta)\ \mathrm{d}v\ \ \mathrm{d}\theta \le 0\ ,
$$

which shows that for each sender i, strategy $\tilde{G}_i$ is a best response to others' strategies $\tilde{G}_{-i}$, and $(\tilde{G}_1, \ldots, \tilde{G}_N)$ is indeed an equilibrium in our game. ■

Putting all the pieces together, finally we can prove Theorem 4.

Proof of Theorem 4. Theorem 4 holds directly by combining Lemmas 18 to 22. ■

D Supplementary Materials in Section 5

D.1 Proof of Lemma 4

Proof of Lemma 4. Given prior F and cost c, we consider a 2-D distribution $G \in \mathcal{G}(F, c)$ (with density g) and the corresponding amortized value distribution K. Let $\tilde{F}(x) = F(x + c)$ for any $x \in [-c, 1-c]$. The lemma amounts to showing that K is an MPC of $\tilde{F}$. The proof proceeds with two steps.

Step 1: Proving $\int_{-c}^{1-c} K(x)\ \mathrm{d}x = \int_{-c}^{1-c} \tilde{F}(x)\ \mathrm{d}x$. Consider any row $\theta \in [-c, 1-c]$, it suffices to show that each row's contribution to both the expected amortized value and the realized value is identical. By Definition 1, we directly have

$$\int_{v=\theta}^{1} g(v,\theta)(v-\theta)\ \mathrm{d}v = c \cdot \int_{v=0}^{1} g(v,\theta)\ \mathrm{d}v\ .$$

For row θ, its contribution to the expected amortized value is $\int_{v=0}^{1} \min\{v,\theta\} g(v,\theta)\ \mathrm{d}v$, while its contribution to the realized value equals $\int_{v=0}^{1}(v-c)g(v,\theta)\ \mathrm{d}v$. The difference between these two contributions for each row is given by:

$$\begin{aligned}
&\int_{v=0}^{1} \min\{v,\theta\} g(v,\theta)\ \mathrm{d}v - \int_{v=0}^{1} (v-c)g(v,\theta)\ \mathrm{d}v \\
&= \int_{v=0}^{\theta} v g(v,\theta)\ \mathrm{d}v + \theta \int_{v=\theta}^{1} g(v,\theta)\ \mathrm{d}v - \int_{v=0}^{1} (v-c)g(v,\theta)\ \mathrm{d}v \\
&= c\int_{v=0}^{1} g(v,\theta)\ \mathrm{d}v - \int_{v=\theta}^{1} g(v,\theta)(v-\theta)\ \mathrm{d}v = 0\ .
\end{aligned}$$

Thus, we complete the proof by showing that

$$\begin{aligned}
&\int_{-c}^{1-c} K(x)\ \mathrm{d}x - \int_{-c}^{1-c} \tilde{F}(x)\ \mathrm{d}x \\
&= \int_{\theta=-c}^{1-c} \left(\int_{v=0}^{1} \min\{v,\theta\} g(v,\theta)\ \mathrm{d}v - \int_{v=0}^{1} (v-c)g(v,\theta)\ \mathrm{d}v \right)\ \mathrm{d}\theta \\
&= \int_{\theta=-c}^{1-c} \left(c\int_{v=0}^{1} g(v,\theta)\ \mathrm{d}v - \int_{v=\theta}^{1} g(v,\theta)(v-\theta)\ \mathrm{d}v \right)\ \mathrm{d}\theta = 0\ .
\end{aligned}$$

Step 2: Proving $\int_{-c}^{t} K(x)\ \mathrm{d}x \le \int_{-c}^{t} \tilde{F}(x)\ \mathrm{d}x$ for any $t \in [-c, 1-c]$. We divide the proof into two parts. First, we show that for any $t \in [-c, 0]$, it holds $\int_{-c}^{t} K(x)\ \mathrm{d}x \le \int_{-c}^{t} \tilde{F}(x)\ \mathrm{d}x$. By using integration by parts, we obtain that

$$\int_{-c}^{t} K(x)\ \mathrm{d}x = \int_{\theta=-c}^{t} \int_{v=0}^{1} (t-\theta) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ ,$$

and

$$\int_{-c}^{t} \tilde{F}(x)\ \mathrm{d}x = \int_{0}^{t+c} F(x)\ \mathrm{d}x = \int_{v=0}^{t+c} \int_{\theta=-c}^{1-c} (t+c-v) g(v,\theta)\ \mathrm{d}\theta\ \mathrm{d}v\ .$$

Besides, according to the Fubini's Theorem, we can swap the order of integration, that is

$$\int_{-c}^{t} \tilde{F}(x)\ \mathrm{d}x = \int_{\theta=-c}^{1-c} \int_{v=0}^{t+c} (t+c-v) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

By Definition 1, for any row $\theta \in [-c, t]$, we have that

$$\int_{v=0}^{1} \theta g(v,\theta)\ \mathrm{d}v = \int_{0}^{1} (v-c) g(v,\theta)\ \mathrm{d}v\ .$$

After integrating both sides of the equation at θ from $-c$ to t, we achieve that

$$\int_{\theta=-c}^{t}\int_{v=0}^{1}(-\theta)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_{\theta=-c}^{t}\int_{v=0}^{1}(c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

Add $\int_{\theta=-c}^{t}\int_{v=0}^{1} tg(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ to both sides of the equation, we have

$$\int_{\theta=-c}^{t}\int_{v=0}^{1}(t-\theta)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_{\theta=-c}^{t}\int_{v=0}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

Thus, we complete the proof by showing that

$$\begin{aligned}\int_{-c}^{t}\tilde{F}(x)\ \mathrm{d}x - \int_{-c}^{t}K(x)\ \mathrm{d}x &= \int_{\theta=0}^{1-c}\int_{v=0}^{t+c}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta - \int_{\theta=-c}^{t}\int_{v=0}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta \\ &= \int_{\theta=t}^{1-c}\int_{v=0}^{t+c}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta - \int_{\theta=-c}^{t}\int_{v=t+c}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta \geq 0\ ,\end{aligned}$$

since $\int_{\theta=t}^{1-c}\int_{v=0}^{t+c}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ is greater than or equal to 0, while $\int_{\theta=-c}^{t}\int_{v=t+c}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ is less than or equal to 0.

Next, we show that for any $t \in [0, 1-c]$, it holds $\int_{-c}^{t} K(x)\ \mathrm{d}x \leq \int_{-c}^{t}\tilde{F}(x)\ \mathrm{d}x$. By using integration by parts, we obtain that

$$\int_{-c}^{t}K(x)\,\mathrm{d}x = \int_{\theta=-c}^{0}\int_{v=0}^{1}(t-\theta)g(v,\theta)\,\mathrm{d}v\,\mathrm{d}\theta + \int_{\theta=0}^{t}\int_{v=\theta}^{1}(t-\theta)g(v,\theta)\,\mathrm{d}v\,\mathrm{d}\theta + \int_{\theta=0}^{t}\int_{v=0}^{\theta}(t-v)g(v,\theta)\,\mathrm{d}v\,\mathrm{d}\theta\ ,$$

and

$$\int_{-c}^{t}\tilde{F}(x)\ \mathrm{d}x = \int_{0}^{t+c}F(x)\ \mathrm{d}x = \int_{v=0}^{t+c}\int_{\theta=-c}^{1-c}(t+c-v)g(v,\theta)\ \mathrm{d}\theta\ \mathrm{d}v\ .$$

Besides, according to the Fubini's Theorem, we can swap the order of integration, that is

$$\int_{-c}^{t}\tilde{F}(x)\ \mathrm{d}x = \int_{\theta=-c}^{1-c}\int_{v=0}^{t+c}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

By Definition 1, for any row $\theta \in [-c, 0]$, we have

$$\int_{v=0}^{1}\theta g(v,\theta)\ \mathrm{d}v = \int_{v=0}^{1}(v-c)g(v,\theta)\ \mathrm{d}v\ .$$

After integrating both sides of the equation at θ from $-c$ to 0, we achieve that

$$\int_{\theta=-c}^{0}\int_{v=0}^{1}(-\theta)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_{\theta=-c}^{0}\int_{v=0}^{1}(c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

Add $\int_{\theta=-c}^{t}\int_{v=0}^{1} tg(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ to both sides of the equation, then we have

$$\int_{\theta=-c}^{0}\int_{v=0}^{1}(t-\theta)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta = \int_{\theta=-c}^{0}\int_{v=0}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta\ .$$

By Definition 1, for any row $\theta \in [0,t]$, we have

$$\int_{v=\theta}^{1} \theta g(v,\theta)\ \mathrm{d}v = \int_{v=\theta}^{1} v g(v,\theta)\ \mathrm{d}v - \int_{v=0}^{1} c g(v,\theta)\ \mathrm{d}v\ .$$

After integrating both sides of the equation at θ from 0 to t, we achieve that

$$\int_{\theta=0}^{t}\int_{v=\theta}^{1} (-\theta) g(v,\theta)\ \mathrm{d}v = \int_{\theta=0}^{t}\int_{v=\theta}^{1} (-v) g(v,\theta)\ \mathrm{d}v + \int_{\theta=0}^{t}\int_{v=0}^{1} c g(v,\theta)\ \mathrm{d}v\ .$$

Add $\int_{\theta=0}^{t}\int_{v=0}^{1} t g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ to both sides of the equation, then we have

$$\int_{\theta=0}^{t}\int_{v=0}^{1} (t-\theta) g(v,\theta)\ \mathrm{d}v = \int_{\theta=0}^{t}\int_{v=\theta}^{1} (t-v+c) g(v,\theta)\ \mathrm{d}v + \int_{\theta=0}^{t}\int_{v=0}^{\theta} c g(v,\theta)\ \mathrm{d}v\ .$$

Thus, we complete the proof by showing that

$$\begin{aligned}\int_{-c}^{t} \tilde{F}(x)\ \mathrm{d}x - \int_{-c}^{t} K(x)\ \mathrm{d}x &= \int_{\theta=-c}^{1-c}\int_{v=0}^{t+c} (t+c-v) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta - \int_{\theta=-c}^{t}\int_{v=0}^{1} (t+c-v) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta \\ &= \int_{\theta=t}^{1-c}\int_{v=0}^{t+c} (t+c-v) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta - \int_{\theta=0}^{t}\int_{v=t+c}^{1} (t+c-v) g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta \geq 0\ ,\end{aligned}$$

since obviously, $\int_{\theta=t}^{1-c}\int_{v=0}^{t+c}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ is greater than or equal to 0, while $\int_{\theta=0}^{t}\int_{v=t+c}^{1}(t+c-v)g(v,\theta)\ \mathrm{d}v\ \mathrm{d}\theta$ is less than or equal to 0. ■

D.2 Proof of Theorem 5

Proof of Theorem 5. When all senders adopt the full-revelation strategy, it is easy to check that the corresponding strategy profile induced is a symmetric equilibrium. We now show that this is the unique symmetric equilibrium.

Since we consider symmetric equilibria, we omit the subscripts for senders. For the sake of contradiction, suppose G^* is another symmetric equilibrium strategy that does not fully reveal the value. Let K denote the distribution of the amortized value induced by the 2-D distribution G^*. By Lemma 4, we know that K shifted by c is a MPC of F, i.e.,

$$\int_{-c}^{t} K(x)\ \mathrm{d}x \leq \int_{0}^{t+c} F(x)\ \mathrm{d}x, \quad \forall t \in [-c, 1-c]\ .$$

Besides, we have $K(-c) = F(0) = 0,\ K(1-c) = F(1) = 1$, and

$$\int_{-c}^{1-c} K(x)\ \mathrm{d}x = \int_{0}^{1} F(x)\ \mathrm{d}x\ .$$

Let $u(v)$ denote the contribution of value $v \in [c,1]$ to the sender's expected utility. Then, we can rewrite the expected utility of the sender as $\int_c^1 u(v)f(v)\ \mathrm{d}v$. Under the fact that strategy G^* forms a symmetric equilibrium, function K must be continuous over $[-c, 1-c]$; otherwise, each sender has a profitable deviation by spreading the index where the mass is. Under this fact, we have the corresponding function $\phi(x) = K^{N-1}(x)$ for any $x \in [-c, 1-c]$. Each sender can always choose to send no information, thereby obtaining an expected utility of $\int_c^1 K^{N-1}(v-c)f(v)\ \mathrm{d}v$. Therefore, each sender is guaranteed to achieve at least an expected utility of $\int_c^1 K^{N-1}(v-c)f(v)\ \mathrm{d}v$, that is

$$
\begin{aligned}
\int_c^1 u(v)f(v)\ \mathrm{d}v \geq \int_c^1 K^{N-1}(v-c)f(v)\ \mathrm{d}v &= \int_c^1 f(v)\ \mathrm{d}\left(\int_{-c}^{v-c} K^{N-1}(t)\ \mathrm{d}t\right) \\
&= \left[f(v)\int_{-c}^{v-c} K^{N-1}(t)\ \mathrm{d}t\right]\Bigg|_{v=c}^{1} - \int_c^1 \int_{-c}^{v-c} K^{N-1}(t)\ \mathrm{d}t\ \mathrm{d}f(v) \\
&= \left[f(v)\int_{c}^{v} F^{N-1}(t)\ \mathrm{d}t\right]\Bigg|_{v=c}^{1} - \int_c^1 \int_{-c}^{v-c} K^{N-1}(t)\ \mathrm{d}t\ \mathrm{d}f(v)\ .
\end{aligned}
$$

Since $\int_{-c}^{v-c} K^{N-1}(t)\ \mathrm{d}t \leq \int_c^v F^{N-1}(t)\ \mathrm{d}t$ for any $v \in [c,1]$ and the density of the prior is non-decreasing over $[c,1]$, we have

$$
\int_c^1 u(v)f(v)\ \mathrm{d}v \geq \left[f(v)\int_{c}^{v} F^{N-1}(t)\ \mathrm{d}t\right]\Bigg|_{v=c}^{1} - \int_c^1 \int_{c}^{v} F^{N-1}(t)\ \mathrm{d}t\ \mathrm{d}f(v) = \int_c^1 F^{N-1}(v)\ \mathrm{d}F(v) = \frac{1}{N}\ .
$$

In any symmetric equilibrium of the N-senders game, each sender can obtain at most an expected utility of $1/N$. Combining these two facts, we obtain that $\int_c^1 u(v)f(v)\ \mathrm{d}v = 1/N$. Therefore, all the inequalities above must hold with equality, where we have

$$
\int_{-c}^{t} K(x)\ \mathrm{d}x = \int_0^{t+c} F(x)\ \mathrm{d}x, \quad \forall t \in [-c, 1-c]\ .
$$

This fact is equivalent to the fact that the symmetric equilibrium strategy G^* is indeed induced by the full-revelation strategy, which forms a contradiction with the assumption. Thus, we have proved that the symmetric equilibrium induced by the full-revelation strategy is the unique symmetric equilibrium. ■

D.3 Proof of Theorem 6

Proof of Theorem 6. Denote the shifted prior by $\tilde{F}(x) := F(x+c)$ for $x \in [-c, 1-c]$ (with density $\tilde{f}$). By Lemma 4, we know that the amortized value distribution of any feasible strategy forms an MPC of the function $\tilde{F}$ over $[-c, 1-c]$. We divide this proof into three steps. Here we provide the proof of the non-degenerate case where $\theta_1' = \theta_2 < 1-c$.

Step 1: There uniquely exist $\theta_1 \in (-c, 0)$ and $\theta_2 \in (0, 1-c]$ such that $\frac{\tilde{F}(\theta_1)}{-\theta_1} = \frac{\tilde{F}(\theta_2)-\tilde{F}(\theta_1)}{\theta_2}$, and $K \in \mathsf{MPC}(\tilde{F})$ over $[-c, 1-c]$. For any $\theta_1 \in (-c, 0)$, we can construct the following linear function:

$$
L_{\theta_1}(\theta) = \frac{\tilde{F}(\theta_1)}{-\theta_1}(\theta - \theta_1)\ , \quad \forall \theta \in [\theta_1, 1-c]\ .
$$

There exists an $\epsilon_1 \in (0, c)$ such that for any $\theta_1 \in (-c, -c+\epsilon)$, it holds that $L_{\theta_1}(\theta) \leq \tilde{F}(\theta)$ for any $\theta \in [\theta_1, 1-c]$. There exists an $\epsilon_2 \in (0, c)$ such that for any $\theta_1 \in (-\epsilon_2, 0)$, the function L_{θ_1} has at least one intersection point with function $\tilde{F}$. Since $\tilde{F}(\theta_1)$ and $\tilde{F}(\theta_1)/-\theta_1$ are both strictly increasing in θ_1 over $(-c, 0)$, we know that there uniquely exists a $\underline{\theta}_1 \in (-c, 0)$ such that $L_{\underline{\theta}_1}(\theta) \leq \tilde{F}(\theta)$ for any $\theta \in [\underline{\theta}_1, 1-c]$, and function $L_{\underline{\theta}_1}$ and function $\tilde{F}$ are tangent at some points. Furthermore, there uniquely exists a $\overline{\theta}_1 \in (\underline{\theta}_1, 0)$ such that $L_{\underline{\theta}_1}(1-c) = \tilde{F}(1-c)$. Thus for any $\theta_1 \in (\underline{\theta}_1, 0)$, function L_{θ_1} and function $\tilde{F}$ have at least two intersection points within the interval $[\theta_1, 1-c]$. For any $\theta_1 \in (\underline{\theta}_1, 0)$, we define $\theta_2 = \max\{\theta \in [0, 1-c] | L_{\theta_1}(\theta) = \tilde{F}(\theta)\}$ and $\theta_3 = \min\{\theta \in [0, 1-c] | L_{\theta_1}(\theta) = \tilde{F}(\theta)\}$. Using the three new parameters, we can split the integral difference from θ_1 to θ_2 into two parts:

$$\int_{\theta_1}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta = \left(\int_{\theta_3}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_3}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta\right) - \left(\int_{\theta_1}^{\theta_3} L_{\theta_1}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_3} \tilde{F}(\theta)\, \mathrm{d}\theta\right) .$$

There exists an $\epsilon_3 \in (0, \overline{\theta}_1 - \underline{\theta}_1)$ such that $\int_{\theta_1}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta < 0$ holds for any $\theta_1 \in (\underline{\theta}_1, \underline{\theta}_1 + \epsilon_3)$. There exists an $\epsilon_4 \in (0, \overline{\theta}_1 - \underline{\theta}_1)$ such that $\int_{\theta_1}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta > 0$ holds for any $\theta_1 \in (\underline{\theta}_1, \underline{\theta}_1 + \epsilon_4)$. Besides, the first part $\int_{\theta_3}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_3}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta$ is decreasing at θ_1 over $(\underline{\theta}_1, 0)$, while the second part $\int_{\theta_1}^{\theta_3} L_{\theta_1}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_3} \tilde{F}(\theta)\, \mathrm{d}\theta$ is increasing at θ_1 over $(\underline{\theta}_1, 0)$. So we know that there uniquely exists a $\theta_1 \in (\underline{\theta}_1, \overline{\theta}_1)$ such that $\int_{\theta_1}^{\theta_2} \tilde{F}(\theta)\, \mathrm{d}\theta - \int_{\theta_1}^{\theta_2} L_{\theta_1}(\theta)\, \mathrm{d}\theta = 0$. Finally, we prove that such a pair of parameters (θ_1, θ_2) exists and is unique.

Step 2: Given θ_1 and θ_2 determined in Step 1, there exists a 2-D distribution that induces the hinge-shaped amortized value distribution. We prove this by a construction. We divide this part of the proof into two cases, based on the shape of K, the amortized value distribution.

Case 1: $K(\theta_1') < 1$. In this case, $\theta_2 = \theta_1' < 1 - c$. Values in the $[0, \theta_1 + c]$ and $[\theta_2 - c, 1]$ are fully revealed, so that K and $\tilde{F}$ coincide on these intervals. We are to pool values in $[\theta_1 + c, \theta_2 + c]$ to produce amortized values uniformly distributed over $[0, \theta_2]$. Next, we show how to make such poolings.

Based on the structure of the function K, the slope over $[0, \theta_2]$ is denoted by $\rho = \frac{\tilde{F}(\theta_2) - \tilde{F}(\theta_1)}{\theta_2}$. Since $K \in \mathsf{MPC}(\tilde{F})$, it holds that $\tilde{f}(0) < \rho$ and $\tilde{f}(\theta_2) > \rho$. Besides, the density $\tilde{f}$ is increasing over $[0, \theta_2]$. So there exists a row $\theta^* \in (0, \theta_2)$ such that $\tilde{f}(\theta^*) = K(\theta^*) = \rho$. We need to ensure the index equation is preserved while pooling the values from $[\theta_1 + c, \theta_2 + c]$ into indices $[0, \theta_2]$, with the density of each row equal to ρ. We divide the density of each value $v \in [\theta^* + c, \theta_2 + c]$ into two parts: we keep a portion of size ρ unpooled to individually form an index $v - c$, while the remaining density will be pooled together with lower values. In details, first we let $\mathrm{supp}(G_{\cdot|\theta}) = \{\theta + c\}$ and $g(\theta + c, \theta) = \rho$ for any $\theta \in (\theta^*, \theta_2)$. Then, we show how to pool the values from $[\theta_1 + c, \theta^* + c]$ and $[\theta^* + c, \theta_2 + c]$ together to form indices over $[0, \theta^*]$. We define functions $\alpha : [0, \theta^*] \to [0, 1]$ and $\beta : [0, \theta^*] \to [0, 1]$. We assume that the support of the conditional distribution G_θ is $\{\alpha(\theta), \beta(\theta)\}$ for each $\theta \in [0, \theta^*]$. We give the initial values of these two functions as $\alpha(0) = \theta_1 + c$ and $\beta(0) = \theta_2 + c$. Given the initial conditions for the functions α and β, we now only need to compute their derivatives to fully characterize these two functions. We consider the following two cases to show how to specifically compute the derivatives of these two functions.

case-1.1: $\alpha(\theta) > \theta$. In this case, we have these two equations:

$$\frac{f(\alpha(\theta))}{\alpha'(\theta)} + \frac{f(\beta(\theta)) - \rho}{-\beta'(\theta)} = \rho \ , \quad (\alpha(\theta) - \theta)\frac{f(\alpha(\theta))}{\alpha'(\theta)\rho} + (\beta(\theta) - \theta)\frac{f(\beta(\theta)) - \rho}{-\beta'(\theta)\rho} = c \ .$$

The first equation means that the sum of densities at points $(\alpha(\theta), \theta)$ and $(\beta(\theta), \theta)$ is equal to ρ. The second equation is exactly the index equation of row θ. Both equations are ordinary differential equations. By combining these two equations, we achieve that:

$$\alpha'(\theta) = \frac{(\beta(\theta) - \alpha(\theta))f(\alpha(\theta))}{\rho(\beta(\theta) - c - \theta)}, \quad \beta'(\theta) = \frac{(\beta(\theta) - \alpha(\theta))(\rho - f(\beta(\theta)))}{\rho(\theta + c - \alpha(\theta))} .$$

case-1.2: $\alpha(\theta) \le \theta$. In this case, we have these two equations:

$$f(\alpha(\theta)) + \frac{f(\beta(\theta)) - \rho}{-\beta'(\theta)} = \rho \ , \quad (\beta(\theta) - \theta)\frac{f(\beta(\theta)) - \rho}{-\beta'(\theta)} = c\left(\frac{f(\alpha(\theta))}{\alpha'(\theta)} + \frac{f(\beta(\theta)) - \rho}{-\beta'(\theta)}\right) .$$

The first equation means that the sum of densities at points $(\alpha(\theta), \theta)$ and $(\beta(\theta), \theta)$ is equal to ρ. The second equation is exactly the index equation of row θ. By combining these two equations, we achieve that:

$$\alpha'(\theta) = \frac{-cf(\alpha(\theta))}{(f(\alpha(\theta)) - \rho)(\beta(\theta) - \theta - c)}, \quad \beta'(\theta) = \frac{f(\beta(\theta)) - \rho}{f(\alpha(\theta)) - \rho} .$$

In both subcases, for each $\theta \in [0, \theta^*]$, since the value $\alpha(\theta)$ and the value $\beta(\theta)$ are pooled together to form the index θ, we have $\alpha(\theta) \le \theta + c \le \beta(\theta)$. Since the prior density f is increasing, we achieve that $\alpha'(\theta) \ge 0$ if $\alpha(\theta) \le \theta^* + c$, and $\beta'(\theta) \le 0$ if $\beta(\theta) \ge \theta^* + c$. In addition to the initial values of both functions, there always exists some $\theta' \in [0, \theta^*]$ such that the function α increases over the interval $[0, \theta']$, while the function β decreases over the same interval. We define the maximal such θ' as $\tilde{\theta}$, i.e., $\tilde{\theta} \triangleq \sup\{\theta \in [0, \theta^*] \mid \alpha'(\theta) \ge 0 \ \& \ \beta'(\theta) \le 0\}$. We next show $\tilde{\theta} = \theta^*$ via contradiction. We assume that $\tilde{\theta} < \theta^*$, then it must hold that $\beta(\tilde{\theta}) = \theta^* + c$ and $\alpha(\tilde{\theta}) = \tilde{\theta} + c$. Since the function K constructed in Step-1 forms an MPC of the function $\tilde{F}$ over $[\theta_1, \theta_2]$, we have

$$F(\theta_2 + c) - F(\theta_1 + c) = \rho \cdot \theta_2 .$$

Besides, based on the construction above, we have

$$\rho \cdot (\theta_2 - \theta^* + \tilde{\theta}) = F(\theta_2 + c) - F(\theta^* + c) + F(\tilde{\theta} + c) - F(\theta_1 + c) .$$

Combining these two equations, we achieve that

$$\rho \cdot (\theta^* - \tilde{\theta}) = F(\theta^* + c) - F(\tilde{\theta} + c) ,$$

which forms a contradiction since $F(\theta^* + c) - F(\tilde{\theta} + c) < \rho \cdot (\theta^* - \tilde{\theta})$. Thus, we have proved that $\tilde{\theta} = \theta^*$. Under this fact, we complete the proof by showing that $\alpha(\theta^*) = \beta(\theta^*) = \theta^* + c$. We also prove it through contradiction. Suppose $\alpha(\theta^*) = a < \theta^* + c < b = \beta(\theta^*)$. Since the constructed function K forms an MPC of the function $\tilde{F}$ over $[\theta_1, \theta_2]$, it holds that

$$F(\theta_2 + c) - F(\theta_1 + c) = \rho \cdot \theta_2 .$$

Besides, based on the construction above, we have

$$\rho \cdot \theta_2 = F(\theta_2 + c) - F(b) + F(a) - F(\theta_1 + c) .$$

These two equations form a contradiction, which makes the assumption invalid. Till now, we have proved that the method we provided above indeed can construct such 2-D distribution that induces the amortized value distribution defined in Theorem 6.

Case 2: There exists $\epsilon > 0$ such that the amortized value distribution K is constant

over $(1-c-\epsilon, 1-c)$**.** The construction and proof follow a similar (and much simpler) approach to Case 1, so we omit them here.

Step 3: Based on the techniques in Section 3, we can verify that the strategy constructed above indeed constitutes a best response to the interim utility function u. Using the Algorithms 1 and 2, we can construct such λ and μ:

$$\lambda(v) = \begin{cases} K(v-c) & \text{if } v \in [0, \theta_1 + c) \cup [c, 1] \ , \\ \rho(v-c) + F(\theta_1 + c) & \text{if } v \in [\theta_1 + c, c) \ , \end{cases}$$

and

$$\mu(\theta) = \begin{cases} \lambda'(\theta + c) & \text{if } \theta \in [0, \theta_1 + c) \cup (\theta_1 + c, \theta_2 + c) \cup (\theta_2 + c, 1] \ , \\ \rho & \text{if } \theta \in \{\theta_1 + c, \theta_2 + c\} \ . \end{cases}$$

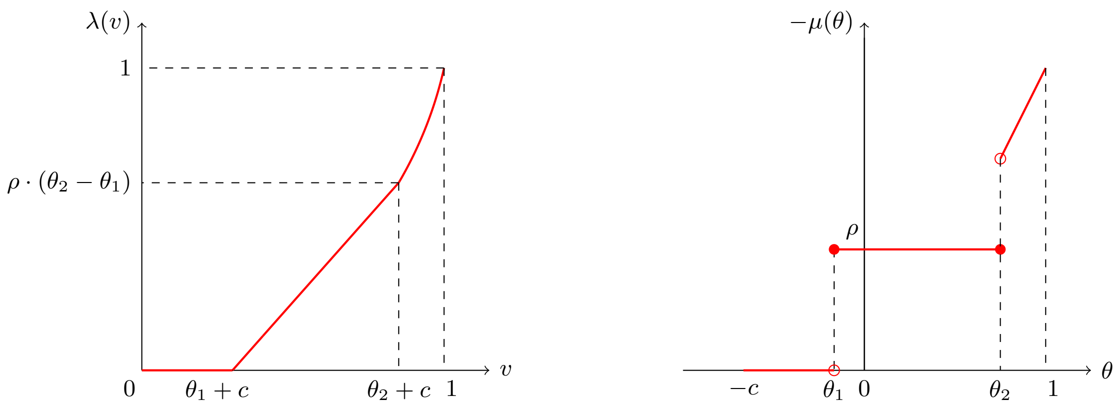

Figure 6: The dual variables constructed by the Recursive Reduction Procedure, based on the symmetric equilibrium constructed in Theorem 6

Based on the formulas of λ and μ, we directly have that for any $(v, \theta) \in \text{supp}(G)$:

$$\lambda(v) = p(v, \theta) - \mu(\theta) q(v, \theta) \ ,$$

which implies that G and (λ, μ) satisfy the Complementary Slackness conditions in Theorem 2. Besides, it also can be verified that such λ and μ are feasible to the corresponding dual problem, that is:

$$\lambda(v) = \max_{\theta \in [-c, 1-c]} p(v, \theta) - \mu(\theta) q(v, \theta) \ , \quad \forall v \in [0, 1] \ ,$$

which makes (λ, μ) feasible to the corresponding dual problem. Combining these two results and by Theorem 3, we know that such G indeed constitutes a best response to the interim utility function u. ■

D.4 Symmetric Equilibrium with Concave Priors

Here we show that the characterization of symmetric equilibrium of concave priors is much more involved than that of convex priors. We consider a two-sender game where $\text{supp}(F) = [c, 1]$, and the prior F is concave over its support. Let $S = \int_0^1 F(x) \, dx$, and we know $S \geq \frac{1}{2}$ since the concavity of the prior.

Different values of the cost c lead to distinct symmetric equilibrium structures, making it challenging to unify all cases under a single amortized value distribution or a generalized 2-D distribution construction. While a complete equilibrium characterization remains an open problem, we present partial results below, specifically detailing the equilibrium structure for cost regimes within certain intervals.

When the cost c is sufficiently large, we make the following construction of a simplest kind of symmetric equilibrium.

Theorem 7. *When the cost $c \geq \frac{2}{3}(1-S)$, there exists a symmetric equilibrium with the amortized value distribution $K(\cdot)$ defined as below:*

$$K(\theta) = \begin{cases} F(\theta + c) & \text{if } \theta \in [-c, 0] \ , \\ \min\left\{\frac{1}{\overline{\theta}}\theta, 1\right\} & \text{if } \theta \in (0, 1-c] \ , \end{cases}$$

where $\overline{\theta}$ is the unique solution to the equation $\int_0^{1-c} \min\left\{\frac{1}{\overline{\theta}}\theta, 1\right\} \, \mathrm{d}\theta = \int_0^{1-c} F(\theta + c) \, \mathrm{d}\theta$.

Proof of Theorem 7. This theorem follows a similar proof of Theorem 6. The construction method can be seen in Figure 7. ■

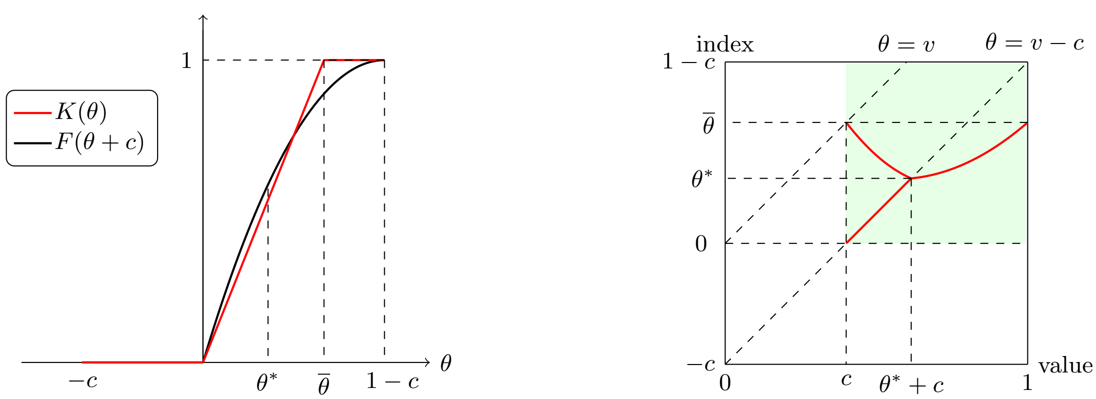

Figure 7: An example of Theorem 7. In the left panel, the red curve represents the amortized value distribution $K(\cdot)$, and the black curve represents the shifted prior. Function $K(\cdot)$ forms an MPC of the shifted prior over $[-c, 1-c]$. In the right panel, the red curves represent the support set of the corresponding 2-D distribution that forms an equilibrium.

This type of symmetric equilibrium may fail to exist when $c < \frac{2}{3}(1-S)$, since in that case it holds $\overline{\theta} > c$, our construction of 2-D distribution may break down halfway. Although this amortized value distribution forms a candidate for equilibrium, but it cannot be guaranteed that there exists a 2-D distribution that induces this amortized value distribution.

D.5 Comparative Statics for Inspection Cost c

We see how the equilibrium strategy varies with the cost c. Recall that values below c generate negative indices if not pooled, thus making zero contribution to the sender's utility. These can be characterized as "potentially useless values". Among these, values above a certain threshold will be pooled with high values (values above c) to form positive indices and contribute to utility, while values below the threshold are the "truly useless values". In the piecewise-linear signaling, the parameter $\theta_1 + c$ represents such a threshold.

As c increases, the amount of potentially useless values grows larger, making it harder for high values to absorb all these potentially useless values. For sufficiently small c, high values can absorb nearly all potentially useless values, causing θ_1 to decrease at c over $[0, 2-\sqrt{2}]$. As c grows larger, high values' ability to absorb the potentially useless values diminishes, leading θ_1 to increase at c over $[2-\sqrt{2}, 1]$. In addition, as c increases, the amount of potentially useless values absorbed by high values also increases, leading the continuous and positive index range shrinks, explaining why θ_2 decreases to zero.

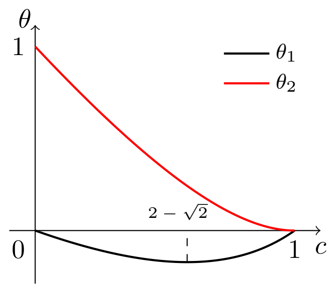

Figure 8: This figure illustrates how the parameters θ_1 and θ_2 change with respect to the cost c.

E Supplementary Materials in Section 6

E.1 Proof of Proposition 1

Proof of Proposition 1. Under the information revelation setting, one search strategy available to the receiver is to ignore the signals and to implement the Index Algorithm based on the original priors. This searching strategy yields the same overall utility as in the no information setting. But this utility is no more than that of the Index Algorithm performed on the posteriors, since the posteriors are more informative than the priors and the agent can better estimate the value based on the posteriors than based on the priors. This proves the first claim.

For the second claim, observe that in the no information setting, the receiver can simulate the obfuscation setting by intentionally ignoring the true value v_i of any inspected box i and instead behaving as if she had observed its posterior mean value. The utility of this simulation is precisely that in the obfuscation setting, which is no more than the utility of the Index Algorithm on the original priors, due to the optimality of Index Algorithm. This proves the second claim. ■

E.2 Proof of Proposition 2 and Conjecture 1

Proof of Proposition 2. When the inspection cost is zero for sender i, the receiver does any inspection for free. Thus, it is optimal for her to inspect all boxes and chooses the most favorable one. An information revealing sender can do nothing, since any information revelation only serves to influence the searching order of the agent. While for an obfuscating sender, he can influence the final choice of the agent through certain obfuscation strategy. Therefore, an obfuscating sender has weakly higher utility.

When the inspection cost is higher than the expectation of the prior distribution, an obfuscating sender always has a negative index regardless of any obfuscation strategy. Thus, he can do nothing in this case, since any box with a negative index will never be inspected under the Index Algorithm. On the other hand, an information-revealing sender can sacrifice some low values and bundle higher values together to form strictly positive indices, thereby ensuring an expected payoff greater than zero. Therefore the latter has a weakly higher utility. ■

We extend Proposition 2 to Conjecture 1, in which we believe that there exists a threshold inspection cost $\hat{c}$ such that, for inspection costs below $\hat{c}$, information obfuscation after inspection is always superior to information revelation before inspection; while for inspection costs above $\hat{c}$, information revelation before inspection is always superior to information obfuscation after inspection. We have confirmed this conclusion through numerical experiments (see Figure 9) and leave the formal proof of this result as future work.

Conjecture 1. *For each sender i, given that the agent's belief distribution over the other boxes and what is observable after inspecting the other boxes remain unchanged, there exists a threshold inspection cost $\hat{c} \in [0, 1]$ such that*

1. *If $0 \le c_1 = ... = c_N \le \hat{c}$, then any strategy of **information obfuscation after inspection**, compared to any strategy of **information revelation before inspection**, provides the sender i with a weakly higher expected utility.*
2. *If $\hat{c} \le c_1 = ... = c_N \le 1$, then any strategy of **information revelation before inspection**, compared to any strategy of **information obfuscation after inspection**, provides the sender i with a strictly higher expected utility.*

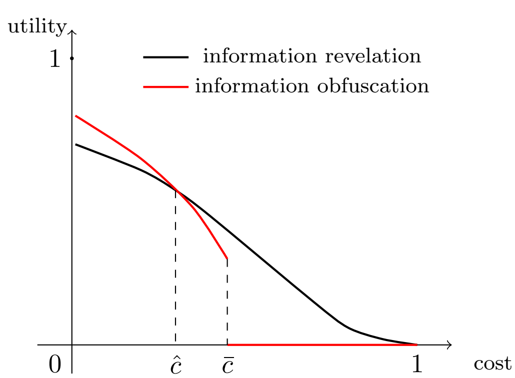

Figure 9: Sender's expected utility under different inspection costs.